Preliminary Examination
Partial Differential Equations
June 2005

Instructions

This is a three-hour examination. You need to solve a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part.**

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

PART ONE

(1) If $n \geq 3$, let $\Gamma(x) = |x|^{2-n}$, when $x \in \mathbf{R^n}$.

(a) Show for an appropriate constant, c_n, that

$$c_n\, f(x) = \int \Gamma(x-y)\, \Delta f(y)\, dy$$

when f is infinitely differentiable with compact support in $\mathbf{R^n}$ ($f \in C_0^\infty(\mathbf{R^n})$). Here Δ denotes the Laplacian, the integral is over $\mathbf{R^n}$, and dy is Lebesgue measure on $\mathbf{R^n}$.

(b) What is c_n?

(2) Let u be bounded and a solution to the heat equation (i.e., $u_t = u_{xx}$) in the rectangle

$$I = (-1,1) \times (0,1) = \{(x,t) : -1 < x < 1, 0 < t < 1\} \subset \mathbf{R^2}$$

with $\lim_{x \to \pm 1} u(x,t) = 0$ for $0 < t < 1$ and $\lim_{t \to 0} u(x,t) = 1$ for $-1 < x < 1$. Assuming that u has a sufficiently smooth extension to the rectangle: $(-2,2) \times (0,1)$,

(a) Show that $\int_0^1 \int_{-1}^1 u_x^2(x,t)dxdt \leq 1$.

(b) What is, $\sup\{u(x,t) : (x,t) \in (-1,1) \times (0,1)\}$? Justify your answer.

(3) Given an open set $U \subset \mathbf{R}^2$, let $C^2(U)$ denote functions with continuous second partials in U. u is said to be a weak solution to the wave equation in U if u is continuous in U and

$$\int_U (\phi_{xx} - \phi_{tt})\, u \, dxdt = 0$$

whenever $\phi \in C_0^\infty(U)$.

(a) Show that if u is a classical solution to the wave equation (i.e. $u \in C^2(U)$ and $u_{xx} = u_{tt}$ pointwise) in U, then u is a weak solution to the wave equation in U.

(b) Show that if $u \in C^2(U)$ and u is a weak solution to the wave equation in U, then u is a classical solution to the wave equation.

(c) Show that if $u(x,t) = g(x-t)$ where g is a continuous function on $\mathbf{R}$, then u is a weak solution to the wave equation in $\mathbf{R}^2$.
Hint: Put $\xi = t + x$, $\eta = t - x$, and change variables in the integral defining a weak solution.

(4) For $(x, y) \in \mathbf{R}^2$ consider the first-order equation $xw_y - yw_x = w$ with initial condition $w(x, 0) = h(x)$ for a given $h \in C_0^\infty(\mathbf{R})$.

(a) Write down the characteristic ordinary differential equations for the above PDE.

(b) Using the method of characteristics and your answer from part (a), find a solution to the above PDE with the prescribed initial values.

(c) What does the method of characteristics say (if anything) about global existence and uniqueness in this problem?

PART TWO

(5) Let $\Omega \subset R^n$ be a bounded domain, and $1 \le p < +\infty$.

(a) State the definition of the Sobolev space $W^{1,p}(\Omega)$.

(b) Show that $u \in W^{1,p}(\Omega)$ if and only if for any open subdomain $U \subset\subset \Omega$, there exists a sequence $\{u_k\} \in C^\infty(U)$ such that $u_k \to u$ in $W^{1,p}(U)$.

(c) Suppose that $F \in C^1(R)$ and F' is bounded. Utilize (b) to prove that for any $u \in W^{1,p}(\Omega)$, we have $F(u) \in W^{1,p}(\Omega)$.

(6) For a bounded domain $\Omega \subset R^n$, assume that $(a_{ij}(x))_{1\leq i,j\leq n} \in L^\infty(\Omega)$ is symmetric. Consider the second order operator

$$L = -\sum_{1\leq i,j\leq n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$$

Answer the following questions.

(a) State the definition of uniform ellipticity of L.

(b) Suppose that L is uniformly elliptic. Then prove the following two inequalities:

$$|B_L[u,v]| \leq C_1\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \ \forall u, v \in H_0^1(\Omega),$$

and

$$B[u,u] \geq C_2\|u\|^2_{H^1(\Omega)}$$

where C_1, C_2 are two positive constants depending only on n and L, and B is the bilinear form associated with L.

(c) Assume that L is uniformly elliptic and $f \in L^2(\Omega)$. State the definition for a $u \in H_0^1(\Omega)$ to be a weak solution of the equation $Lu = f$, in Ω. Is the Dirichlet problem uniquely solvable in $H_0^1(\Omega)$? Provide the reasons (e.g. the name of the theorem) to support your conclusion.

(7) Let L be a uniformly elliptic operator defined as in problem 6. Prove the Caccioppoli inequality: if $u \in H^1(\Omega)$ is a weak solution of $Lu = 0$ in Ω, then, for any ball $B_R \subset \Omega$, one has

$$\int_{B_r} |\nabla u|^2 \leq \frac{C}{(R-r)^2}\int_{B_R} |u|^2, \ \forall 0 < r \leq R$$

where $C > 0$ depends only on n.

(8) For a bounded domain $\Omega \subset R^n$, $f \in L^2(\Omega)$. Let Δ be the standard Laplace operator.

(a) Prove the following Dirichlet principle: $u \in H_0^1(\Omega)$ is a weak solution to: $-\Delta u = f$ in Ω if and only if

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - fu \leq I(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - fv$$

for all $v \in H_0^1(\Omega)$.

(b) Prove the following estimate:

$$\int_\Omega |\nabla u|^2 \leq C\int_\Omega |f|^2$$

for some universal positive constant C. (Hint: You may need to use both the Hölder inequality and the Poincaré inequality)