Preliminary Examination in Partial Differential Equations

2 June 2006

Instructions

This is a three-hour examination. The exam is divided into two parts. You should attempt at least two questions from each part and a total of five questions. Please indicate clearly on your test paper which five questions are to be graded.

Provide complete solutions to each problem and give as much detail as possible. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly the theorems and definitions you are using.

1/06

PART I

1. Consider the Cauchy or initial-value problem,

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0 & \text{in } \mathbf{R}^n \times (0,\infty) \\ u(x,0) = g(x), & x \in \mathbf{R}^n \end{cases}$$

where $g \in C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$. Define u by

$$u(x,t) = \int_{\mathbf{R}^n} \Phi(x-y,t) g(y)\, dy, \qquad x \in \mathbf{R}^n \; t > 0,$$

where $\Phi(x,t)$ is the *fundamental* solution of the heat equation

$$\Phi(x,t) := \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, \qquad x \in \mathbf{R}^n,\ t > 0. \tag{1}$$

(a) Show that we may compute derivative $\partial u/\partial x_1$ by differentiating under the integral.

(b) Show that

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0).$$

2. Consider wave equation in one space dimension, $u_{tt} - c^2 u_{xx} = 0$, where c is a positive constant.

(a) By introducing the variables $\xi = x - ct$ and $\eta = x + ct$, find all smooth solutions of the one-dimensional wave equation in $\mathbf{R}^2$.

(b) Consider the initial-boundary value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x > 0,\ t > 0 \\ u(0,t) = 0, & t > 0 \\ u(x,0) = f(x) & x > 0 \\ u_t(x,0) = 0, & x > 0 \end{cases}$$

Use part a) to find the solution u. You will need to choose an appropriate extension of f to the real line.

(c) Give conditions on f which imply that u is $C^2([0,\infty) \times [0,\infty))$. Pay particular attention to the behavior of f at 0.

3. Suppose that u is in $C^\infty([0,1] \times [0,\infty))$ and is a solution of the equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x,t) \in (0,1) \times (0,\infty) \\ u_t(1,t) = \epsilon u_x(1,t) & t > 0 \\ u(0,t) = 0 & t > 0 \end{cases}$$

1/06

Give a condition on the sign of ϵ which guarantees that the energy

$$E(t) = \frac{1}{2}\int_0^1 |u_x(x,t)|^2 + |u_t(x,t)|^2\, dx$$

is non-increasing.

For this choice of sign, prove that the energy is non-increasing.

4. Use the method of characteristics to find the solution of the initial value problem

$$\begin{cases} u_t + uu_x = 0, & x \in \mathbf{R},\ t > 0 \\ u(x,0) = -x, & x \in \mathbf{R} \end{cases}$$

PART II

5. (a) For a domain $\Omega \subset R^n$, state the definition of the Sobolev space $W^{2,p}(\Omega)$ for $1 \le p < \infty$.

   (b) Prove the following interpolation inequality:

$$\int_{\mathbf{R}^n} |\nabla u|^2\, dx \le \left(\int_{\mathbf{R}^n} |u|^2\, dx\right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^n} |\nabla^2 u|^2\, dx\right)^{\frac{1}{2}}$$

   where $u \in W^{2,2}(\mathbf{R}^n)$. Here, $|\nabla^2 u| = (\sum_{i,j=1}^n |D_i D_j u|^2)^{1/2}$.
   Partial credit will be given for proofs of an estimate of this form with a constant larger than 1.

6. For any symmetric matrix-valued function $(a_{ij}) \in C^0(\Omega, \mathbf{R}^{n\times n})$, let

$$L = -\sum_{ij=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}), \qquad x \in \Omega \tag{2}$$

   (a) State the uniform ellipticity condition for L.

   (b) Suppose that $f \in H^{-1}(\Omega)$. Give the definition of weak solution to the boundary value problem

$$\begin{cases} Lu = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{3}$$

   and then prove that if L is uniformly elliptic, then there exists at most one solution to the boundary value problem (3).

   (c) Use the difference quotient method to prove: If we also assume $(a_{ij}) \in C^1(\Omega, \mathbf{R}^{n\times n})$ and $f \in L^2(\Omega)$, then any weak solution $u \in H^1(\Omega)$ to the boundary value problem (3) is in $H^2_{\text{loc}}(\Omega)$. Moreover, for any open ball $B \subset\subset \Omega$,

$$\|D^2 u\|_{L^2(B)} \le C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}), \tag{2}$$

   where C depends on the dimension, n, the distance from B to the boundary and the coefficients.

7. Let L be the operator (2) and assume L is uniformly elliptic. Prove the Caccioppoli inequality: if $u \in H^1(\Omega)$ is a weak solution to $Lu = 0$ on Ω. Then, for any ball $B_{2R} \subset \Omega$, one has

$$\int_{B_R} |\nabla u|^2 \, dx \le \frac{C}{R^2} \inf_{c \in \mathbf{R}} \int_{B_{2R}} |u - c|^2 \, dx.$$

8. Let L be the operator (2) and assume L is uniformly elliptic. Assume that $u \in H^1(\Omega)$ is a bounded weak solution to

$$Lu = 0, \quad \text{in} \quad \Omega$$

Let $\phi \in C^\infty(R)$ be convex. Set $w = \phi(u)$. Show that $w \in H^1(\Omega)$ and w is a weak subsolution, i.e.

$$B[w, \psi] \le 0, \forall \psi \in H_0^1(\Omega), \text{ with } \psi \ge 0$$

where $B[\cdot, \cdot]$ is the bilinear form associated with L.

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Solutions to selected problems

4. Use the method of characteristics to solve

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = -x \end{cases}$$

Solution. Let τ, ξ, z solve the ODE's:

$$\frac{d}{ds}\tau = 1, \quad \tau(r,0) = 0$$

$$\frac{d}{ds}\xi(r,s) = z \qquad \xi(r,0) = r$$

$$\frac{d}{ds} z(r,s) = 0 \qquad z(r,0) = -r.$$

Solutions are:

$$\tau(r,s) = s$$
$$z(r,s) = -r$$
$$\xi(r,s) = -rs + r.$$

If we define $(\xi(r,s), \tau(r,s)) = (x,t)$, we define $z(r,s) = u(x,t)$

Then

$$0 = \frac{dz}{dr} = \frac{\partial u}{\partial x}\frac{d\xi}{dr} + \frac{\partial u}{\partial t}\frac{d\tau}{dr}$$

$$= 1 u_x + u u_t .$$

Thus u solves the pde.

As $u(x,0) = z(x,0) = -x$

u solves the initial condition.

Solving

$$x = r(1-s)$$
$$t = s$$

gives $r = \frac{x}{1-s} = \frac{x}{1-t}$

Thus $u(x,t) = z(r,s) = -r$

$$= -\frac{x}{1-t}.$$

#5. (a) For a domain $\Omega \subset \mathbb{R}^n$ give the definition of the Sobolev space $W^{2,p}(\Omega)$, $1 \le p < \infty$

Definition. Let u be in $L^1_{loc}(\Omega)$, we say that $g \in L^1_{loc}(\Omega)$ is the weak derivative of u, with respect to x_i,

$$g = \frac{\partial u}{\partial x_i}$$

if for all $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega g \varphi \, dx = -\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx.$$

Definition. We say that $u \in W^{2,p}(\Omega)$ if $D^\alpha u \in L^p(\Omega)$ for all α with $|\alpha| = 0, 1$ or 2.

(b) Prove the interpolation inequality

$$\int_\Omega |\nabla u|^2 dx \le \left(\int |u|^2\right)^{1/2} \left(\int |\nabla^2 u|^2\right)^{1/2}$$

for $u \in W^{2,2}(\mathbb{R}^n)$.

Proof. 1. Suppose $u \in C_0^\infty(\mathbb{R}^n)$

then by Green's identity and Cauchy Schwarz.

$$\int |\nabla u|^2 dx = -\int u \Delta u$$

$$\le \left(\int u^2\right)^{1/2} \left(\int (\Delta u)^2\right)^{1/2}$$

2. If $u \in C_0^\infty(\mathbb{R}^n)$, then

$$\int (\Delta u)^2 dx = \int |\nabla^2 u|^2 dx$$

Proof We integrate by parts twice

$$\int (\Delta u)^2 dx = \sum_{i,j=1}^{n} \int D_i D_i u \, D_j D_j u$$

$$= - \sum \int D_i u \, D_i D_j D_j u$$

$$= + \sum \int D_i D_j u \, D_i D_j u$$

$$= \sum_{i,j=1}^{n} \int |D_i D_j u|^2 dx. \blacksquare$$

3. From parts 1) and 2) we have:

If $u \in C_c^\infty(\mathbb{R}^n)$, then

$$\int |\nabla u|^2 dx \le \left(\int u^2\right)^{1/2} \left(\int |\nabla^2 u|^2\right)^{1/2}.$$

4. The inequality

$$\int |\nabla u|^2 dx \le \left(\int u^2\right)^{1/2} \left(\int |\nabla^2 u|^2\right)^{1/2}.$$

holds for all $u \in W^{2,2}(\mathbb{R}^n)$.

<u>Proof</u> We assume u is in $W^{2,2}(\mathbb{R}^n)$. Regularizing u and cutting off u near infinity, we may find a sequence u_j with $u_j \to u$ in $W^{2,2}(\mathbb{R}^n)$ and $u_j \in C_c^\infty(\mathbb{R}^n)$. As $u_j \to u$ in $W^{2,2}(\mathbb{R}^n)$, we have

$$\lim_{j\to\infty} \int u_j^2 \, dx = \int u^2 \, dx$$

$$\lim_{j\to\infty} \int |\nabla u_j|^2 = \int |\nabla u|^2 \, dx$$

$$\lim_{j\to\infty} \int |\nabla^2 u|^2 dx = \int |\nabla^2 u|^2 dx$$

Thus, the inequality for $u \in W^{2,2}(\mathbb{R}^n)$ follows from step 3, where u is in $C_c^\infty(\mathbb{R}^n)$.