Preliminary Examination on Partial Differential Equations

May 30, 2007

Instructions

This is a three-hour examination. You need to solve a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part**.

Please indicate clearly on your test papers that which five problems are to be graded.

You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

Throughout this exam, Ω denotes a bounded domain in $\mathbb{R}^n$ with smooth boundary.

PART ONE

Problem 1. Let $D = \{x \in \mathbb{R}^n : 1 < |x| < 2\}$ and let $\mathcal{F}$ be the set of all harmonic functions defined on D.

(a) Show that if $y = Ax$ is a rotation of $S = \{x : |x| = 1\}$ and $u \in \mathcal{F}$, then $v \in \mathcal{F}$, where $v(x) = u(Ax)$. Hint: if x, y are viewed as column vectors, then A is an orthonormal matrix.

(b) Show that if $u \in \mathcal{F}$ satisfies $u(x) = u(Ax)$ for all rotations $y = Ax$ of S and $n \geq 3$, then $u(x) = c|x|^{2-n} + d$ for some c, d where c, d are constants.

Problem 2. (a) State and prove Harnack's inequality for positive harmonic functions in $B(0,1) = \{x : |x| < 1\} \subset \mathbb{R}^n$.

(b) Use part (a) to prove that a nonconstant harmonic function in $B(0,1)$ cannot have an absolute maximum at the origin.

Problem 3. Let $(x,t) \in \mathbb{R}^2$ and put

$$\Gamma(x,t) = \begin{cases} (4\pi t)^{-1/2} \exp(-\frac{x^2}{4t}) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Show that $\Gamma(x,t)$ is a fundamental solution to the heat equation in the sense that if ϕ is infinitely differentiable with compact support in $\mathbb{R}^2$, then

$$\phi(x,t) = \iint_{\mathbb{R}^2} \Gamma(x-y, t-s)\{\phi_s(y,s) - \phi_{yy}(y,s)\}\, dyds.$$

You may use the fact that $\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi}$.

Problem 4. Let $u \in C^2(\overline{\Omega} \times [0,\infty))$ solve the heat equation

$$u_t - \Delta u = 0 \quad \text{in } \Omega \times (0,\infty)$$

and $u(x,t) = 0$ on $\partial\Omega \times (0,\infty)$. Prove that

(a) $\int_\Omega |u(x,t)|^2 dx$ is monotonically non-increasing in t,

(b) $\int_\Omega |\nabla_x u(x,t)|^2 dx$ is monotonically non-increasing in t.

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PART TWO

Problem 5. Let $u \in C^2(\mathbb{R}\times[0,\infty))$ solve the initial value problem for the one-dimensional wave equation:

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \quad \text{in } \mathbb{R}\times(0,\infty), \\ u = g, u_t &= h \quad \text{on } \mathbb{R}\times\{t=0\}. \end{aligned}$$

Suppose that g, h have compact support. Set $k(t) = \int_{\mathbb{R}} u_t^2(x,t)\,dx$ and $p(t) = \int_{\mathbb{R}} u_x^2(x,t)dx$. Prove that

(a) $k(t) + p(t)$ is constant in t,

(b) $k(t) = p(t)$ for all large enough t.

In the following three problems,

$$\mathcal{L}u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right)$$

where $a_{ij} = a_{ji} \in L^\infty(\Omega)$. We assume that $\mathcal{L}$ is uniformly elliptic.

Problem 6. Let $f \in L^2(\Omega)$.

(a) State the definition of weak solutions $u \in H_0^1(\Omega)$ to the boundary value problem (BVP) : $\mathcal{L}u = f$ in Ω and $u = 0$ on $\partial\Omega$.

(b) Prove the existence and uniqueness of the weak solution to the (BVP) in part (a). Also show that the solution satisfies

$$\|u\|_{H^1(\Omega)} \le C\,\|f\|_{L^2(\Omega)},$$

where C depends only on $\mathcal{L}$, Ω, and n.

(c) Let u be the weak solution to the (BVP) in part (a). Suppose that $f = 0$ in $B(x_0, r) \subset \Omega$. Show that

$$\int_{B(x_0,r/2)} |\nabla u|^2\,dx \le \frac{C}{r^2}\int_{B(x_0,r)} |u|^2\,dx,$$

where C depends only on $\mathcal{L}$ and n.

Problem 7. Show that

$$\int_\Omega |\nabla u|^2\,dx \le C\left\{\int_\Omega |u|^2\,dx\right\}^{1/2}\left\{\int_\Omega |\nabla\nabla u|^2\,dx\right\}^{1/2},$$

for all $u \in H^2(\Omega)\cap H_0^1(\Omega)$, where C depends only on n.

Problem 8. Assume that $a_{ij} \in C^1(\overline{\Omega})$. Let u be a smooth solution of $\mathcal{L}u = 0$ in Ω. Show that

$$\|\nabla u\|_{L^\infty(\Omega)} \le C\left\{\|\nabla u\|_{L^\infty(\partial\Omega)} + \|u\|_{L^\infty(\partial\Omega)}\right\},$$

where C depends only on $\mathcal{L}$ and n.