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May 28, 2008 **9:00 AM - 12:00 PM**
CB 343

Instructions

This is a three-hour examination. The exam is divided into two parts. You should attempt at least two questions from each part and a total of five questions. Please indicate clearly on your test paper which five questions are to be graded.

Provide complete solutions to each problem and give as much detail as possible. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly the theorems and definitions you are using.

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Part I

1. Let $B(x,r) = \{y : |y-x| < r\}$ whenever $x \in R^n$ and $r > 0$. A function u is said to be strongly subharmonic in $B(0,1)$, if u has continous second partials in $B(0,1)$ and

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} > 0 \text{ at each } x \text{ in } B(0,1).$$

   (a) Show that strongly subharmonic functions in $B(0,1)$ have the following mean value property: If $x \in B(0,1)$ and $0 < r < 1 - |x|$, then

$$u(x) < \frac{1}{\sigma(S)} \int_S u(x + r\omega)\, d\sigma(\omega)$$

   where $S = \{\omega : |\omega| = 1\}$ and σ is surface area on S.

   Hint : Use Green's formula ,

$$\int_{B(x,\rho)} \Delta u\, dx = \rho^{n-1} \int_S \frac{\partial u}{\partial \rho}(x + \rho\omega)\, d\sigma(\omega) \text{ whenever } \bar{B}(x,\rho) \subset B(0,1)\ .$$

   (b) Use (a) to show that u does not have a local maximum in $B(0,1)$.

2. (a) Prove or disprove : For some positive $\delta > 0$ there exists a function $u(x,y)$ harmonic in the half disk, $H = \{(x,y) : y > 0,\ 0 < x^2 + y^2 < \delta^2\} \subset R^2$ and continuous in the closure of H with boundary values :
   $u(x,0) = 0,\ -\delta < x < \delta,$ while $u_y(x,0) = e^{-\frac{1}{x^2}}, -\delta < x < \delta,\ x \neq 0.$

   (b) Does there exist $\delta > 0$ such that $u(x,y) = x - y$ is the unique nonzero solution to the wave equation, $u_{xx} = u_{yy}$, in $O = \{(x,y) : y > x, x^2 + y^2 < \delta^2\}$ which is continuous on the closure of O with boundary values: $u(x,x) = 0,\ u_x(x,x) = -u_y(x,x),\ -\delta < x < \delta$?

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(c) Can the theorem of Cauchy-Kovalevski be used to answer either (a) or (b) ? Give reasons for your answers to (a), (b), and (c).

3. Use the method of characteristics to find a solution $u(x,y)$, $(x,y) \in \mathbb{R}^2$, of the first order equation

$$(y+u)u_x + yu_y = x - y$$

such that

$$u(x,1) = 1 + x$$

for all $x \in \mathbb{R}$. Indicate the domain of the solution.

4. The fundamental solution to the heat equation

$$\Delta u(x,t) = \frac{\partial u}{\partial t}(x,t)$$

is

$$\Gamma(x,t) = (4\pi t)^{-n/2} \exp\left(-|x|^2/4t\right)$$

and has the properties that

$$\int_{\mathbb{R}^n} \Gamma(x,t)\, dx = 1$$

$$\lim_{t \downarrow 0} \int_{|x|>\delta} \Gamma(x,t)\, dt = 0$$

Show that if $f \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then the function

$$v(x,t) = \int_{\mathbb{R}^n} \Gamma(x-y,t) f(y)\, dy$$

obeys

$$\lim_{\substack{(x,t)\to(x^0,0) \\ x\in\mathbb{R}^n,\ t>0}} v(x,t) = f(x^0)$$

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Part II

Throughout we will denote by $\{a_{ij}(x)\}^n_{\substack{i=1\\j=1}}$ bounded and uniformly elliptic coefficients on the domain U and by L the elliptic operator defined in problem 1 relative to this sequence.

1. Suppose that u is a weak solution of

$$Lu \equiv \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} = 0$$

in $\mathbb{R}^n$, i.e. U is all of $\mathbb{R}^n$. Show that if $\int_{\mathbb{R}^n} |u(x)|^2\,dx < \infty$ then $u \equiv 0$. [Hint: Use the test function $u(x)\eta(x/R)^2$, where $\eta \in C_c^\infty(B(0,1))$, estimate and then let $R \to \infty$.]

2. Suppose $u = u(x,t)$ is a weak solution to the problem

$$u_t + Lu = f(x,t), \text{ for } (x,t) \in U \times [0,T],$$
$$u(x,t) = 0, \text{ for } (x,t) \in \partial U \times [0,T],$$
$$u(x,0) = 0, \text{ for } x \in U.$$

Assume that $\int_0^T \int_U f(x,t)^2\,dxdt$ is finite for the given $U \subset \mathbb{R}^n$. Show

$$\lim_{t \to 0+} \frac{1}{t} \int_U u(x,t)^2\,dx = 0.$$

[Hint: Get estimates on $\eta(t) = \int_U u(x,t)^2\,dx$.]

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3. Let $u \in H_0^1(U)$ be a weak solution to

$$(*) \ Lu + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x) u = 0$$

where $\{b_i(x)\}_{i=1}^n$ and $c(x)$ are bounded smooth coefficients on the bounded domain $U \subset \mathbb{R}^n$.

(a) Prove the Gårding-type estimate

$$B(u,u) \geq \alpha ||u||^2_{H_0^1(U)} - k_0 ||u||^2_{L^2(U)}$$

for two positive constants α and k_0. Here $B(u,v)$ is the standard bilinear form associated with the partial differential equation given by $(*)$.

(b) Show that if $c(x) - \operatorname{div} \vec{b}(x) \geq 0$ on U then k_0 can be taken to be zero. Why is this significant?

4. (a) Define the concept(s): the domain $U \subset \mathbb{R}^n$ has a C^∞-boundary/real analytic boundary ∂U.

(b) Show that the unit disk in $\mathbb{R}^2$, $x^2 + y^2 \leq 1$, has a C^∞-boundary/a real analytic-boundary.

(c) Give an example of a curve in the plane that is C^∞, but not real analytic, at least at one point on the curve.