Preliminary Examination in Partial Differential Equations
June 3 , 2010

Instructions This is a three-hour examination.The exam is divided into two parts. You need to solve a total of five problems. You must do at least two problems from each part. Please indicate clearly on your test papers which five problems are to be graded.

You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

Notation. Throughout this preliminary exam n denotes a positive integer > 2 and $\nabla\cdot$, D, Δ denote respectively the divergence, gradient, and Laplace operators. Also Euclidean n space is denoted by $\mathbf{R}^n$ while $B(x,r) = \{y \in \mathbf{R}^n : |y-x| < r\}$.

PART I

(1) (a) If $u \in C_0^2(\mathbf{R}^n)$ show that

$$u(0) = -\frac{1}{(n-2)\omega_n} \int_{\mathbf{R}^n} |y|^{2-n} \Delta u(y)\, dy$$

where ω_n is the surface area of the unit sphere in $\mathbf{R}^n$.

(b) Let $n > 2$ be a positive integer and let f be continuous on $\mathbf{R}^n$ with compact support in $B(0,r)$. Prove directly (i.e., do not quote a theorem) that if

$$h(x) = \int_{\mathbf{R}^n} |x-y|^{2-n} f(y) dy, x \in \mathbf{R}^n,$$

then

$$|h(x)| \le c \frac{r^n}{(r+|x|)^{n-2}} \|f\|_{L^\infty(\mathbf{R}^n)}$$

whenever $x \in \mathbf{R}^n$, where c depends only on n.

(2) Let U be a bounded C^1 domain and suppose that $u \in C^1(\bar{U}) \cap C^2(U)$ is a solution to the boundary value problem:

$$\begin{cases} \Delta u - 3u = f \text{ in } U \\ u + \frac{\partial u}{\partial \nu} = g \text{ on } \partial U \end{cases}$$

where $\nu = \nu(x)$ is the outer unit normal to ∂U at $x \in \partial U$. Show that u is unique. **Hint:** Use the identity, $\nabla \cdot (uDu) = |Du|^2 + u\Delta u$.

(3) Given the initial value problem for the wave equation in $\mathbf{R} \times (0, \infty) \subset \mathbf{R}^2$. and $f \in C^2(\mathbf{R}^2)$:

$$u_{tt}(x,t) - u_{xx}(x,t) = f(x,t),$$
$$u(x,0) = u_t(x,0) = 0.$$

(a) Consider the change of variables

$$\zeta = x + t,$$
$$\eta = x - t.$$

Note that this transformation maps the line $t = 0$ into the line $\zeta = \eta$. Show that this mapping has Jacobian 2 and that

$$\frac{\partial}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right),$$
$$\frac{\partial}{\partial \eta} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right).$$

Also, show that the triangle T_1 with vertices $(x-t,0)$, $(x+t,0)$ and (x,t) in the xt plane is mapped to the triangle T_2 with vertices $(x-t, x-t)$, $(x+t, x+t)$, and $(x+t, x-t)$ in the $\zeta\eta$-plane.

(b) Show that this initial value problem is equivalent to the problem

$$\begin{array}{lll} (\alpha) & 4\dfrac{\partial^2}{\partial\eta\partial\zeta}U(\zeta,\eta) & = -F(\zeta,\eta) \\ (\beta) & U(\zeta,\zeta) & = 0 \\ (\gamma) & U_\zeta(\zeta,\zeta) - U_\eta(\zeta,\zeta) & = 0 \end{array}$$

where

$$U(\zeta,\eta) = u\left(\frac{1}{2}(\zeta+\eta), \frac{1}{2}(\zeta-\eta)\right),$$
$$F(\zeta,\eta) = f\left(\frac{1}{2}(\zeta+\eta), \frac{1}{2}(\zeta-\eta)\right).$$

(c) By integrating equation (α) over the triangle T_2 in the $\zeta\eta$-plane and using the boundary conditions $(\beta) - (\gamma)$, show that

$$U(\zeta,\eta) = \frac{1}{4}\int\int_{T_2} F(\zeta',\eta')\, d\zeta'\, d\eta' = \frac{1}{4}\int_{x-t}^{x+t}\left(\int_{x-t}^{\zeta'} F(\zeta',\eta') d\eta'\right) d\zeta'$$

Conclude that

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,s)\, dy\, ds.$$

(4) Let $b = (b_1, \dots, b_n)$ be a point in $\mathbf{R}^n$.

(a) Use the method of characteristics to find a C^1 solution in $\mathbf{R}^n \times [0, \infty)$ to the linear transport problem

$$\begin{cases} u_t + b \cdot Du = e^t \text{ in } \mathbf{R}^n \times (0, \infty) \\ u(x, 0) = |x|^2 \text{ for } x \in \mathbf{R}^n . \end{cases}$$

(b) Show uniqueness of the solution that you found in (a).

PART II

(5) Given that if $u \in W^{1,1}(\mathbf{R}^n)$, then

$$\|u\|_{L^{n/(n-1)}(\mathbf{R}^n)} \leq c\|Du\|_{L^1(\mathbf{R}^n)}$$

where c depends only on n. Use this fact to show that if $1 < p < n$, $p^* = \frac{np}{n-p}$ and $u \in W^{1,p}(\mathbf{R}^n)$, then

$$\|u\|_{L^{p^*}(\mathbf{R}^n)} \leq c'\|Du\|_{L^p(\mathbf{R}^n)}$$

where c' is a positive constant depending only on p, n. You may assume that $C_0^\infty(\mathbf{R}^n)$ functions are dense in $W^{1,p}(\mathbf{R}^n)$.

(6) Let $U = B(0,1) \setminus \bar{B}(0, 1/2)$ and suppose that $0 < u \in C^2(U) \cap C^1(\bar{U})$ is a positive classical solution to

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j}(x) = 0 \text{ in } U \text{ with } u \equiv 0 \text{ on } \partial B(0,1).$$

In this display $(a_{ij}(x))$ are continuous in $\bar{U}$ and

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, x \in \bar{U},$$

for some $0 < \lambda, \Lambda < \infty$, and all $\xi \in \mathbf{R}^n$. If $m = \min_{\partial B(0,1/2)} u$ show that

$$|Du(x)| \geq m/c \text{ for } x \in \partial B(0,1),$$

where $c \geq 1$ depends only on n, λ, Λ. You may assume the maximum principle for subsolutions to L.

(7) (a) State the Rellich - Kondrachov Theorem for a bounded C^1 domain $U \subset \mathbf{R}^n$ relative to $W^{1,2}(U)$.

(b) Use the theorem you stated in (a) to prove Poincaré's inequality for $W^{1,2}(U)$:

$$\int_U (v - v_U)^2 \, dx \leq C \int_U |Dv|^2 dx$$

where v_U denotes the average of v on U and $C \geq 1$ is a constant independent of $v \in W^{1,2}(U)$.

(8) Given $g \in W^{1,2}(B(0,1))$ we say that $v \in W^{1,2}(B(0,1))$ is a weak solution to

$$\hat{L}v = - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial v}{\partial x_i}(x) \right) = 0 \text{ in } B(0,1),$$

with $v = g$ on $\partial B(0,1)$, provided that $v - g \in W_0^{1,2}(B(0,1))$ and

$$\int_{B(0,1)} \sum_{i,j=1}^{n} a_{ij}(x) v_{x_i} \zeta_{x_j} \, dx = 0$$

whenever $\zeta \in W_0^{1,2}(B(0,1))$. In this display $(a_{ij}(x))$ is a symmetric matrix with measurable coefficients, satisfying

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, x \in \bar{B}(0,1),$$

for some $0 < \lambda, \Lambda < \infty$, and all $\xi \in \mathbf{R}^n$.

(a) Show that if v is a weak solution to $\hat{L}v = 0$ in $B(0,1)$ with $v = g$ on $\partial B(0,1)$, then

$$\int_{B(0,1)} |Dv|^2 dx \leq (\Lambda / \lambda) \int_{B(0,1)} |Dg|^2 dx .$$

(b) Given $g \in W^{1,2}(B(0,1))$, let $F = \{ f \in W^{1,2}(B(0,1)) \text{ with } f - g \in W_0^{1,2}(B(0,1)) \}$ and put $I(f) = \sum_{i,j=1}^{n} \int_{B(0,1)} a_{ij}(x) \, f_{x_i} f_{x_j} \, dx.$

Show that if there exists $v \in F$ with $I(v) = \min\{ I(f) : f \in F \}$, then necessarily v is a weak solution to $\hat{L}v = 0$ in $B(0,1)$ with boundary values g.