Preliminary Examination in Partial Differential Equations
June 6, 2011

Instructions This is a three-hour examination. The exam is divided into two parts. You need to solve a total of five problems. You must do at least two problems from each part. Please indicate clearly on your test papers which five problems are to be graded.

You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

Notation. Euclidean n space is denoted by $\mathbf{R}^n$ while $B(x,r) = \{y \in \mathbf{R}^n : |y - x| < r\}$.

PART I

(1) Let u be a harmonic function in the connected open set Ω.
- (a) State the mean value property for u.
- (b) State the strong maximum principle for u.
- (c) Use the mean value property to prove the strong maximum principle for u.

(2) Let u be harmonic in $B(0,1)$ and let ∇u denote the gradient of u. Define, for $0 < r < 1$,

$$H = H(r) = \int_{\partial B(0,r)} u^2 d\sigma, \quad \text{and} \quad D = D(r) = \int_{B(0,r)} |\nabla u|^2 dx.$$

- (a) Show that

$$\frac{dH}{dr} = \frac{n-1}{r} H(r) + 2D(r).$$

- (b) Show that

$$\text{div}\{|\nabla u|^2 x\} = 2\text{div}[(x \cdot \nabla u)\nabla u] + (n-2)|\nabla u|^2.$$

- (c) Use (b) to show that

$$\frac{dD}{dr} = 2\int_{\partial B(0,r)} \left|\frac{\partial u}{\partial r}\right|^2 d\sigma + \frac{n-2}{r} D(r).$$

- (d) Use (a) and (c) to show that

$$\frac{d}{dr}\left\{\frac{rD(r)}{H(r)}\right\} \geq 0.$$

(3) Let Ω be a bounded smooth domain in $\mathbf{R}^n$. Suppose that $u \in C^2(\bar{\Omega} \times [0,\infty))$ is a solution to the initial-Neumann problem

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = g(x), \quad x \in \Omega, \end{cases}$$

where ν denotes the outward unit normal to $\partial\Omega$, and g is continuously differentiable on $\bar{\Omega}$ with $\frac{\partial g}{\partial \nu} = 0$ on $\partial\Omega$. Show that for any $0 < T < \infty$,

$$\int_\Omega |u(x,T)|^2 dx + 2\int_0^T \int_\Omega |\nabla u(x,t)|^2 dx dt = \int_\Omega |g(x)|^2 dx.$$

In this problem Δu and ∇u denote the Laplacian and gradient of u in the space variable x, only.

(4) (a) Use the method of characteristics to find a solution, $u(x_1,x_2)$ satisfying

$$x_1 u_{x_1} - x_2 u_{x_2} = u^3 \text{ with } u(1,x_2) = x_2^2,\ x_2 > 0.$$

(b) Let g be an infinitely differentiable function on $\mathbf{R}$ with compact support and $g(0) = 0$. Show that $u(x_1,x_2) = g(x_1 - x_2)$ is a solution to $u_{x_1} + u_{x_2} = 0$ with $u(x_1,x_1) \equiv 0$ for $-\infty < x_1 < \infty$. Can any of these solutions be found by the method of characteristics? Give a reason for your answer.

PART II

(5) (a) For fixed $p, 1 \le p < \infty$, give the definition of $W_0^{1,p}(B(0,1))$ when $B(0,1) \subset \mathbf{R}^n$.

(b) Fix $\alpha, 0 < \alpha < 1$, and let

$$u(x) = (1 - |x|^2)^\alpha, x \in B(0,1).$$

Prove that $u \in W_0^{1,p}(B(0,1))$ provided $1 \le p < 1/(1-\alpha)$. You may assume that the usual partial derivatives of u are weak or distributional derivatives of u.

(6) (a) State the Rellich-Kondrachov Compactness Theorem for a bounded C^1 domain $\Omega \subset \mathbf{R}^n$.

(b) Apply part (a) to prove the following version of Poincaré's inequality: If $\Omega = B(0,1)$, $u \in H^1(\Omega)$, and $\Big|\{x \in \Omega :\ u(x) = 0\}\Big| \geq \alpha > 0$, then there exists a constant C depending only on n and α, such that

$$\int_\Omega u^2\,dx \leq C \int_\Omega |\nabla u|^2\,dx.$$

(7) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and $a_{ij} \in L^\infty(\Omega)$ for $1 \leq i, j \leq n$. Define the second order differential operator L by

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big).$$

(a) Define what is meant by u is a weak solution to

$$\begin{cases} Lu = f & \text{for } f \in L^2(\Omega) \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

(b) Give conditions on L and briefly discuss how these conditions can be used (along with the Lax Milgram theorem), to obtain a unique weak solution u to the PDE problem in (a).

(c) If u, L, f are as in (a), (b), prove that

$$\int_U |\nabla u|^2\,dx \leq C\left(\int_U [f^2 + u^2]\,dx\right),$$

where C depends only on n and the ellipticity constants.

(8) Let $u(x,t) \in C^\infty(\overline{\Omega} \times [0, +\infty))$ and let $\Delta u, \nabla u$ be the Laplacian, gradient of u in x, only. Suppose that

$$\begin{cases} u_t - \Delta u = \lambda(t) u & \text{in } \Omega \times (0, +\infty) \\ u = 0 & \text{on } \partial\Omega \times [0, +\infty) \\ u = g & \text{on } \Omega \times \{t = 0\}, \end{cases}$$

where $0 \neq g \in C_0^\infty(\Omega)$ is a given function. Prove that if

$$\lambda(t) = \frac{\int_\Omega |\nabla u(x,t)|^2\,dx}{\int_\Omega g^2\,dx},\ t \geq 0,$$

then $\displaystyle\int_\Omega u^2(x,t)\,dx \equiv \int_\Omega g^2\,dx$ for all $t \geq 0$.

(Hint: consider $\frac{d}{dt}\left(\int_\Omega u^2\,dx - \int_\Omega g^2\,dx\right)$)