Preliminary Examination in Partial Differential Equations

3 June 2013

Instructions

This is a three-hour examination. The exam is divided into two parts. You should attempt at least two questions from each part and a total of five questions. Please indicate clearly on your test paper which five questions are to be graded.

Provide complete solutions to each problem and give as much detail as possible. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly the theorems and definitions you are using.

PART I

Throughout the exam, $B(x, r)$ denotes the open ball in $\mathbf{R}^n$ with center x and radius r. We use D_s to denote the partial derivative with respect to a variable s. We use $\Delta = \sum_{j=1}^n D_{x_j}^2$ to denote the Laplacian and $D = (D_{x_1}, \ldots, D_{x_n})$ to denote the gradient on $\mathbf{R}^n$.

1. Use d'Alembert's formula to find a solution of

$$\begin{cases} D_t^2 u - D_x^2 u = 0, & (x,t) \in \mathbf{R}^2 \\ u(x,0) = \cos(2x), & x \in \mathbf{R} \\ u_t(x,0) = 0, & x \in \mathbf{R}. \end{cases}$$

2. Suppose that u is in $C^2(\bar{B}(0,1))$ and solves

$$\begin{cases} \Delta u = f, & \text{in } B(0,1) \\ u = 0 & \text{on } \partial B(0,1). \end{cases}$$

Show that

$$|u(x)| \leq \frac{1}{2n} \sup |f|, \qquad x \in B(0,1).$$

Hint: Use that $\Delta(|x|^2 - 1) = 2n$.

3. Let $u \in C(\bar{B}(0,1))$ be given by the Poisson integral on the unit ball

$$u(x) = \frac{1 - |x|^2}{\omega_{n-1}} \int_{\partial B(0,1)} \frac{u(y')}{|x - y'|^n} \, d\sigma(y'), \qquad x \in B(0,1).$$

Here ω_{n-1} is the surface area of the unit ball and $d\sigma$ is surface measure. Prove that if $u \geq 0$ in $\bar{B}(0,1)$, then

$$\sup_{B(0,1/2)} u \leq C \inf_{B(0,1/2)} u$$

where the constant C depends only on the dimension. Your proof should give an explicit value for C.

4. Let u be a smooth solution of the wave equation $D_t^2 u - \Delta u = 0$ in $\mathbf{R}^{n+1}$ and suppose that

$$u(x,0) = D_t u(x,0) = 0, \qquad |x| \leq 1.$$

Prove that $u = 0$ in the set $\{(x,t) : |x| \leq 1 - t, \ 0 < t < 1\}$.

Part II

5. Let $B^+(0,r) = B(0,r) \cap \{x : x_n > 0\}$ for $r > 0$. Let $u \in W^{1,2}(B^+(0,1))$ and define $\hat{u}(x)$ by

$$\hat{u}(x) = \begin{cases} u(x), & x \in B^+(0,1) \\ u(x', -x_n), & x \in B(0,1) \cap \{x_n < 0\}. \end{cases}$$

   (a) Show that $\hat{u} \in W^{1,2}(B(0,1))$.

   (b) Use part a) to show that there exists a sequence of functions $\{u_m\} \subset C^\infty(\bar{B}^+(0,1/2))$ which converges to $u|_{B^+(0,1/2)}$ in the norm of $W^{1,2}(B^+(0,1/2))$.

6. The Newtonian capacity of a compact set $K \subset B(0,1)$, denoted by $N(K)$, is defined as follows. Let $\mathcal{F} = \mathcal{F}(K)$ be the collection of functions ϕ in $W^{1,2}(\mathbf{R}^n)$ with $\phi = 1$ on K and $\phi = 0$ in $\mathbf{R}^n \setminus B(0,2)$. Then

$$N(K) = \inf_{\phi \in \mathcal{F}} \int_{\mathbf{R}^n} |D\phi|^2 \, dx.$$

   (a) Show that the Newtonian capacity of the closed unit ball $N(\bar{B}(0,1))$ is finite.

   (b) Use the Poincaré inequality to show that the the Newtonian capacity of the closed unit ball $N(\bar{B}(0,1))$ is positive.

7. Prove the Gagliardo-Nirenberg-Sobolev inequality on $\mathbf{R}^3$: If u is in the Sobolev space $W^{1,1}(\mathbf{R}^3)$, then $\|u\|_{L^{3/2}(\mathbf{R}^2)} \leq C\|Du\|_{L^1(\mathbf{R}^3)}$.

8. Consider the elliptic operator L on given by

$$Lu = -\sum_{j,k=1}^{3} D_{x_j} a_{jk}(x) D_{x_k} u + \sum_{j=1}^{3} b_j D_{x_j} u.$$

   Suppose that L acts on functions in $W^{1,2}(B(0,1))$ with $B(0,1) \subset \mathbf{R}^3$.

   (a) Write the bilinear form B corresponding to the operator L.

   (b) Assume the coefficients $\{a_{jk} : j,k = 1,\dots,3\}$ are in $L^\infty(B(0,1))$ and the coefficients $\{b_j : j = 1,\dots,3\}$ lie in $L^3(B(0,1))$. Show that the form B is bounded on the Sobolev space $W_0^{1,2}(B(0,1))$.

   (c) Suppose that in addition to being bounded, the coefficients $\{a_{jk}\}$ also satisfy the ellipticity condition,

$$\sum_{j,k=1}^{3} a_{jk}\xi_j\xi_k \geq \lambda|\xi|^2, \qquad x \in B(0,1), \xi \in \mathbf{R}^n$$

   for some $\lambda > 0$. Show that the form B will be coercive on $W_0^{1,2}(B(0,1))$ if each of the coefficient functions b_j is small in $L^3(B(0,1))$.

Russell Brown, Peter Hislop, John Lewis–May 20, 2013