Preliminary Examination in Partial Differential Equations

4 June 2014

Instructions

This is a three-hour examination. The exam is divided into two parts. You should attempt at least two problems from each part and a total of five problems. Please indicate clearly on your test paper which five problems are to be graded.

Provide complete solutions to each problem and give as much detail as possible. More weight will be given to a complete solution to one problem than to solutions of the easy bits from two different problems. Indicate clearly the theorems and definitions you are using.

PART I

1. Let B_1 denote the open unit ball with center 0 in $\mathbf{R}^n$. Let $f \in C(B_1)$ such that $|f(x)| \leq 1$ for $x \in B_1$. Assume $u \in C^2(B_1) \cap C(\overline{B_1})$ and solves

$$\begin{cases} -\Delta u = f & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases}$$

Henceforth $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian operator on $\mathbf{R}^n$. Prove that

$$-\frac{1}{2n}(1 - |x|^2) \leq u(x) \leq \frac{1}{2n}(1 - |x|^2), \ \forall\, x \in B.$$

2. Given that f is a bounded and continuous function in $\mathbf{R}^n$, define $u : \mathbf{R}^n \times (0, +\infty) \to \mathbf{R}$ by

$$u(x,t) = \int_{\mathbf{R}^n} \Gamma(x-y,t) f(y)\, dy, \ (x,t) \in \mathbf{R}^n \times (0, +\infty),$$

where

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \ (x,t) \in \mathbf{R}^n \times (0,+\infty),$$

is the fundamental solution of the heat equation. Prove that

(a) for any $x \in \mathbf{R}^n$,

$$\lim_{t \to 0+} u(x,t) = f(x).$$

(b) there exists a constant $C > 0$ depending on n and $\|f\|_{L^\infty(\mathbf{R}^n)}$ such that

$$\left|Du(x,t)\right| \leq \frac{C}{\sqrt{t}}, \ \forall\, (x,t) \in \mathbf{R}^n \times (0, +\infty).$$

3. Use d'Alembert's formula to find the solution of the wave equation in one spatial dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x,t) \in \mathbf{R}^2, \\ u(x,0) = 0 & x \in \mathbf{R}, \\ u_t(x,0) = \sin(2x) & x \in \mathbf{R}. \end{cases}$$

4. Use the method of characteristics to solve the first order PDE:

$$\begin{cases} u_{x_1} u_{x_2} = u, & \text{in } U, \\ u = x_2^2, & \text{on } \Gamma, \end{cases}$$

where $U = \left\{x \in \mathbf{R}^2 : \ x_1 > 0\right\}$ and $\Gamma = \partial U = \left\{x \in \mathbf{R}^2 : \ x_1 = 0\right\}$.

Part II

1. Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and consider the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

   (a) State what it means for $u \in H_0^1(\Omega)$ to be a weak solution of the boundary value problem.

   (b) Show that if f is in $L^{\frac{2n}{n+2}}(\Omega)$, then there exists a weak solution $u \in H_0^1(\Omega)$ to the boundary value problem.
   (**Hint**: First verify that f defines an element of $H^{-1}(\Omega)$ and then apply the Lax-Milgram theorem).

2. (a) Suppose that u lies in the Sobolev space $W^{1,2}(\mathbf{R}^n)$. For $1 \leq j \leq n$, let $e_j = (0, ..., 1, ..., 0) \in \mathbf{R}^n$ be the j^{th} unit vector, and for $h \in \mathbf{R} \setminus \{0\}$ let

$$D_j^h u(x) = \frac{u(x + he_j) - u(x)}{h}$$

   denote the difference quotient in the j^{th} direction. Show that

$$\left\| D_j^h u \right\|_{L^2(\mathbf{R}^n)} \leq \left\| \frac{\partial u}{\partial x_j} \right\|_{L^2(\mathbf{R}^n)}.$$

   (b) Suppose that $u \in L^2(\mathbf{R}^n)$ and there exists a constant $A > 0$ so that

$$\left\| D_j^h u \right\|_{L^2(\mathbf{R}^n)} \leq A$$

   for all h. Show that the weak derivative $\frac{\partial u}{\partial x_j}$ exists in $\mathbf{R}^n$ and

$$\left\| \frac{\partial u}{\partial x_j} \right\|_{L^2(\mathbf{R}^n)} \leq A.$$

3. Let Ω be a bounded open set and consider the Rayleigh quotient

$$\mathcal{R}[\phi] = \frac{\int_\Omega |D\phi(x)|^2 \, dx}{\int_\Omega \phi(x)^2 \, dx}$$

for nonzero, real valued functions $\phi \in W_0^{1,2}(\Omega)$.

(a) Show that there exists a positive lower bound for $\mathcal{R}$, i.e., there exists $c > 0$ such that

$$\mathcal{R}[\phi] \geq c \text{ for all } \phi \in W_0^{1,2}(\Omega) \setminus \{0\}.$$

(b) If $0 \neq u \in W_0^{1,2}(\Omega)$ is a minimum for $\mathcal{R}[\cdot]$ over $W_0^{1,2}(\Omega) \setminus \{0\}$, i.e.,

$$\mathcal{R}[u] = \min \left\{ \mathcal{R}[\phi] : \ \phi \in W_0^{1,2}(\Omega),\ \phi \neq 0 \right\},$$

then there exists $\lambda > 0$ so that u is a weak solution of

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

(**Hint:** If ϕ is in $W_0^{1,2}(\Omega)$, consider the function of a real variable $f(t) = \mathcal{R}[u + t\phi]$ and show that $f'(0) = 0$.)

4. For $0 < T \leq +\infty$ and an open subset $\Omega \subset \mathbf{R}^n$, let $(a_{ij})_{1 \leq 1, j \leq n} \in L^\infty(\Omega \times (0,T))$. Consider the operator

$$\mathcal{L}u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right).$$

(a) State the definition that $\mathcal{L}$ is a uniformly elliptic operator.

(b) State the definition that $u \in L^2(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ is a weak solution of

$$\partial_t u - \mathcal{L}u = 0 \text{ in } \Omega \times (0,T),$$

and then prove the following parabolic Caccipolli type inequality: there exists a constant $C > 0$ such that for any $(x_0, t_0) \in \Omega \times (0,T)$, it holds

$$\sup_{t_0 - R^2 \leq t \leq t_0} \int_{B_R(x_0)} |u(x,t)|^2 \, dx + \int_{t_0 - R^2}^{t_0} \int_{B_R(x_0)} |Du(x,t)|^2 \, dxdt \leq \frac{C}{R^2} \int_{t_0 - 4R^2}^{t_0} \int_{B_{2R}(x_0)} |u(x,t)|^2 \, dxdt,$$

provided $B_{2R}(x_0) \times [t_0 - 4R^2, t_0] \subset \Omega \times (0,T)$.