Preliminary Examination in Partial Differential Equations

Instructions **June 2015**

This is a three-hour examination. You are to work a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part.**

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

PART ONE

Problem 1. Suppose that $u \in C^2(\overline{B(0,1)})$ and solves

$$\Delta u = f \quad \text{in } B(0,1),$$
$$u = 0 \quad \text{on } \partial B(0,1)$$

for some $f \in C^0(\overline{B(0,1)})$. Prove that

$$|u(x)| \leq \frac{1}{2d} \sup |f| \quad \text{for any } x \in B(0,1).$$

Hint. Use the fact that $\Delta(|x|^2 - 1) = 2d$.

Problem 2. Suppose that $u \in C^2(B(0,r)) \cap C^0(\overline{B(0,r)})$ is harmonic in $B(0,r)$ and that $u|_{\partial B(0,r)} = g$. Recall the representation formula

$$u(x) = \frac{1}{d\,\alpha(d) r} \int_{\partial B(0,r)} \frac{r^2 - |x|^2}{|x-y|^d} g(y)\, d\sigma(y).$$

Show that for any $x \in B(0,r)$,

$$r^{d-2} \frac{r - |x|}{(r + |x|)^{d-1}} u(0) \leq u(x) \leq r^{d-2} \frac{r + |x|}{(r - |x|)^{d-1}} u(0).$$

Problem 3. Let f be a bounded, continuous function on $\mathbb{R}^d$ and let u be the solution of the initial value problem,

$$\begin{cases} D_t u - \Delta u = 0, & (x,t) \in \mathbb{R}^d \times (0,\infty) \\ u(\cdot, 0) = f. \end{cases}$$

Recall that u is given by the expression

$$u(x,t) = \int_{\mathbb{R}^d} \Gamma(x-y,t) f(y)\, dy,$$

where $\Gamma(x,t)$ is the fundamental solution of the heat equation and is defined by

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{d/2}} \exp\left(\frac{-|x-y|^2}{4t}\right).$$

(1) Show that

$$\lim_{t \to 0+} u(x,t) = f(x).$$

You should state the properties of $\Gamma(x,t)$ that you use, but you do not need to provide proofs.

(2) Show that you may find a constant C so that

$$\sup_{(x,t)\in\mathbb{R}^d\times(0,\infty)} t|D_t u(x,t)| \le C \sup|f|.$$

Problem 4. Suppose that $u : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 solution of the equation

$$D_{x_1}D_{x_2}u - D_{x_2}^2 u = 0.$$

Show that there are functions F and G so that

$$u(x_1, x_2) = F(x_1) + G(x_1 + x_2).$$

PART TWO

Problem 5. Assume $F : \mathbb{R} \to \mathbb{R}$ is C^1, with F' bounded. Let Ω be a bounded domain in $\mathbb{R}^d$ and $u \in W^{1,p}(\Omega)$ for some $1 < p < \infty$. Let $w = F(u)$. Show that

$$\frac{\partial w}{\partial x_i} = F'(u)\frac{\partial u}{\partial x_i} \quad \text{and} \quad w \in W^{1,p}(\Omega).$$

Problem 6. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^d$. Let

$$\mathcal{L} = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right),$$

where $a_{ij} \in L^\infty(\Omega)$.

(1) What does it mean if $\mathcal{L}$ is said to be uniformly elliptic?

(2) Let $f \in L^2(\Omega)$. State the definition for a function $u \in H_0^1(\Omega)$ to be a weak solution of the Dirichlet problem:

$$\begin{cases} \mathcal{L}(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

(3) Prove that there exists a unique weak solution to the Dirichlet problem in part (2).

Problem 7. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^d$. Let

$$\mathcal{L} = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$$

be a uniformly elliptic operator on Ω.

(1) Let $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$. Assume that

$$\int_\Omega f\,dx + \int_{\partial\Omega} g\,d\sigma = 0.$$

State the definition for a function $u \in H^1(\Omega)$ to be a weak solution of the Neumann problem:

$$\begin{cases} \mathcal{L}(u) = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases}$$

where $\frac{\partial u}{\partial \nu} = \sum_{i,j} n_i a_{ij} \frac{\partial u}{\partial x_j}$ denotes the conormal derivative associated with $\mathcal{L}$.

(2) Prove that there exists a weak solution to the Neumann problem in part (1). Also show that the solutions is unique, up to a constant.

Problem 8. Let

$$\mathcal{L} = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$$

be a uniformly elliptic operator on Ω. Suppose $u \in H^1(\Omega)$ is a weak solution of $\mathcal{L}(u) = 0$ in Ω. Show that

$$\int_{B(x_0,r)} |\nabla u(x)|^2\,dx \le \frac{C}{r^2}\int_{B(x_0,2r)} |u(x)|^2\,dx$$

for any $B(x_0, 2r) \subset \Omega$, where C depends only on d and the ellipticity constant of $\mathcal{L}$. This is the Caccioppoli inequality.