Preliminary Examination
in Partial Differential Equations

June 2016

Instructions

This is a three-hour examination. You are to work a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part**.

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

PART ONE

Problem 1. Suppose that $f \in C(\overline{B(0,1)})$, and $a, g \in C(\partial B(0,1))$, and $a(x) > 0$ for all $x \in \partial B(0,1)$. Show that the Robin boundary value problem

$$\begin{aligned} \Delta u = f \qquad & \text{in } B(0,1), \\ au + \partial_\nu u = g \qquad & \text{on } \partial B(0,1) \end{aligned}$$

has at most one solution $u \in C^2(\overline{B(0,1)})$. Here, $B(0,1)$ denoted the ball of radius one centered at the origin in $\mathbb{R}^n$, and ∂_ν is the outward normal derivative.

Problem 2. Let $n \geq 3$, and let $\Phi(x)$ be the fundamental solution for the Laplacian in $\mathbb{R}^n$:

$$\Phi(x) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}},$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$. Suppose $f \in C_c^\infty(\mathbb{R}^n)$ and let

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy.$$

Show, without using distributions, that

$$-\Delta u = f.$$

Problem 3. Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and $T > 0$, and suppose $u \in C^2(\overline{\Omega} \times [0,T])$ is a solution of the heat equation

$$u_t = \Delta u \text{ in } \Omega \times (0,T).$$

Let $\Gamma = (\Omega \times \{0\}) \cup (\partial\Omega \times [0,T])$ and let $U_T = \Omega \times [0,T]$. Show that

$$\max_{\Gamma} u = \max_{\overline{U}_T} u.$$

Use this fact to show that the problem

$$\begin{aligned} u_t - \Delta u &= f \text{ in } \Omega \times (0,T) \\ u|_\Gamma &= g \end{aligned}$$

has at most one solution $u \in C^2(\overline{\Omega} \times [0,T])$. One possible way to do the first part is to define $u_\epsilon = u - \epsilon t$, for $\epsilon > 0$, and show that u_ϵ cannot attain its maximum over $\overline{U}_T$ at a point in U_T.

Problem 4. Suppose that φ and ψ are C^2 functions on $\mathbb{R}$, and $u(x,t)$ is a C^2 solution of the equation

$$2u_{xt} - u_{tt} = 0 \text{ on } \mathbb{R} \times \mathbb{R}_+$$

with initial conditions at $t = 0$ given by

$$u(x,0) = \varphi(x) \text{ and } u_t(x,0) = \psi(x).$$

Find a formula for $u(x,t)$ in terms of φ and ψ.

PART TWO

Problem 5. Suppose that $u \in W^{1,p}(\mathbb{R})$, for some $p \in (1,\infty)$. Show directly that

$$\|u\|_{C^{0,1-1/p}(\mathbb{R})} \leq \|u\|_{W^{1,p}(\mathbb{R})}.$$

Problem 6. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$, with $n \geq 3$. Consider the boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

(1) State what it means for $u \in H_0^1(\Omega)$ to be a weak solution of this boundary value problem.
(2) Show that if $f \in L^{\frac{2n}{n+2}}(\Omega)$, then there exists a weak solution $u \in H_0^1(\Omega)$ to this boundary value problem.

Problem 7. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^d$. Let $a_{ij} \in C^1(\overline{\Omega})$ be such that the operator

$$\mathcal{L} = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$$

is a uniformly elliptic operator on Ω, and suppose $u \in C_c^3(\Omega)$ is a weak solution to

$$\mathcal{L}u = f$$

for some $f \in L^2(\Omega)$. Show that there exists some constant $C > 0$ depending only on the a_{ij} such that

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}).$$

Problem 8. Let Ω be a bounded smooth domain in $\mathbb{R}^n$.

(1) State the Rellich-Kondrachov theorem for Ω.

(2) Show that there exists a constant $C > 0$ depending only on Ω, such that for all $v \in W^{1,2}(\Omega)$,

$$\int_\Omega (v - v_\Omega)^2 \, dx \leq C \int_\Omega |Dv|^2 \, dx$$

where v_Ω is the average of v over Ω. Hint: if this is false, then for each $k \in \mathbb{N}$, there exists v^k with

$$\int_\Omega (v^k - v^k_\Omega)^2 \, dx \geq k \int_\Omega |Dv^k|^2 \, dx$$