Preliminary Examination in Partial Differential Equations

June 2018

Instructions

This is a three-hour examination. You are to work a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part**.

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

PART ONE

Problem 1. Let $B(0,r) = \{x \in \mathbb{R}^d : |x| < r\}$. Recall Poisson's formula for $B(0,r)$: if $g \in C^0(\partial B(0,r))$ and

$$\begin{cases} \Delta u = 0 & \text{in } B(0,r), \\ u = g & \text{on } \partial B(0,r), \end{cases}$$

then for $x \in B(0,r)$,

$$u(x) = \frac{r^2 - |x|^2}{d\alpha(d) r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^d}\, d\sigma(y),$$

where $\alpha(d)$ denotes the volume of the unit ball in $\mathbb{R}^d$. Using Poisson's formula, show that

$$r^{d-2} \frac{r - |x|}{(r+|x|)^{d-1}} u(0) \le u(x) \le r^{d-2} \frac{r+|x|}{(r-|x|)^{d-1}} u(0)$$

Problem 2.

1. (1) Prove that a uniformly convergent sequence of harmonic functions is harmonic. You may use the fact that harmonicity is equivalent to the mean value property.
2. (2) Prove *Harnack's convergence theorem*: if $\{u_n\}$ is a monotone increasing sequence of harmnoic functions on a domain Ω, and there is a $y \in \Omega$ so that $\{u_n(y)\}$ is a bounded sequence, then $\{u_n\}$ converges uniformly on any bounded subdomain to a harmonic function.

Problem 3. Find all solutions of the one-dimensional heat equation, $u_t = u_{xx}$ of the form

$$u(x,t) = v\left(\frac{x}{\sqrt{t}}\right).$$

Hint: Show that v satisfies a linear ordinary differential equation. Your answer will involve an anti-derivative that cannot be evaluated in elementary terms.

Problem 4. Suppose that u is a smooth function on $\mathbf{R}^d \times [0,\infty)$ and that u satisfies

$$\begin{cases} u_{tt} - \Delta u = 0, & t \ge 0, \\ u(x,0) = u_t(x,0) = 0, & |x| \le 1. \end{cases}$$

Show that $u(x,t) = 0$ for $\{(x,t) : |x| \le 1 - t\}$.

Hint: Consider the expression

$$\int_{|x|<1-t} \left\{ |\nabla u(x,t)|^2 + u_t^2(x,t) \right\} dx.$$

PART TWO

In the following we assume that the $d \times d$ matrix $\big(a_{ij}(x)\big)$ satisfies the uniform ellipticity condition,

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and for any } \xi \in \mathbb{R}^d, \tag{1}$$

where $\lambda > 0$. The domain Ω in $\mathbb{R}^d$ is assumed to be bounded and its boundary is smooth.

Problem 5.

(1) Give the definition of the weak derivatives of a function u in Ω.
(2) State the Sobolev inequalities for functions in $W^{1,p}(\Omega)$ for $1 < p < d$, and for $d < p < \infty$. Give an example of a function that is in $W^{1,d}(\Omega)$, but not in $L^\infty(\Omega)$.
(3) Use the Fourier transform to show that if $u \in H^s(\mathbb{R}^d)$ for $s > \frac{d}{2}$, then $u \in L^\infty(\mathbb{R}^d)$, and that

$$\|u\|_{L^\infty(\mathbb{R}^d)} \leq C \, \|f\|_{H^s(\mathbb{R}^d)},$$

where C depends only on d and s.

Problem 6. Let

$$\mathcal{L} = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$$

be a uniformly elliptic operator on Ω.

(1) Let $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$. Assume that

$$\int_\Omega f\,dx + \int_{\partial\Omega} g\,d\sigma = 0.$$

State the definition for a function $u \in H^1(\Omega)$ to be a weak solution of the Neumann problem:

$$\begin{cases} \mathcal{L}(u) = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases}$$

where

$$\frac{\partial u}{\partial \nu} = \sum_{i,j} n_i a_{ij} \frac{\partial u}{\partial x_j}$$

denotes the conormal derivative associated with $\mathcal{L}$.

(2) Prove that there exists a weak solution to the Neumann problem in part (1). Also show that the solution is unique, up to a constant.

Problem 7. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of

$$-\sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial u}{\partial x_i} = 0 \quad \text{ in } \Omega,$$

where $(a_{ij}(x))$ is a symmetric matrix satisfying the ellipticity condition (1), and $a_{ij}, b_i \in C(\overline{\Omega})$. Show that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Problem 8. Let

$$\mathcal{L} = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i} + c(x),$$

where $a_{ij}, b_i, c \in L^\infty(\Omega)$, and a_{ij} satisfies the uniform ellipticity condition (1). Let $u \in H^1(\Omega)$ be a weak solution of $\mathcal{L}(u) = f$ in Ω, where $f \in L^2(\Omega)$. Show that

$$\|\nabla u\|_{L^2(V)} \leq C\big\{\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\big\},$$

where V is an open subset of Ω such that $\overline{V} \subset \Omega$, and C depends only on d, λ, V, Ω, $\|a_{ij}\|_{L^\infty(\Omega)}$, $\|b_i\|_{L^\infty(\Omega)}$, and $\|c\|_{L^\infty(\Omega)}$.

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