Preliminary Examination in Partial Differential Equations

June 2019

Instructions

This is a three-hour examination. You are to work a total of five problems. The exam is divided into two parts. **You must do at least two problems from each part.**

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

PART ONE

Problem 1.

Let Ω be a open set in $\mathbb{R}^d$ and $u \in C^2(\Omega)$. We say u is subharmonic in Ω if

$$-\Delta u \leq 0 \quad \text{in } \Omega.$$

(i) Prove that if u is subharmonic in Ω, then

$$u(x) \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} u\,dy \quad \text{for all } B(x,r) \subset\subset \Omega.$$

(ii) Assume that Ω is bounded. Prove that if u is subharmonic in Ω and $u \in C(\overline{\Omega})$, then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Problem 2.

(i) Let u be a harmonic function in $\mathbb{R}^d$. Suppose that there exist constant C and nonnegative integer k such that

$$|u(x)| \leq C(|x|+1)^k \qquad \text{for all } x \in \mathbb{R}^d.$$

Show that u is a polynomial of degree k or less.

(ii) Let $\mathcal{H}_k$ denote the vector space of all polynomials $P(x)$ with real coefficients and of degree k or less such that $\Delta P = 0$ in $\mathbb{R}^d$. Find the dimension of $\mathcal{H}_k$ in the case $k = d = 2$.

Problem 3.

Consider the heat equation

$$(1) \qquad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^d, \end{cases}$$

where $g : \mathbb{R}^d \to \mathbb{R}$ is continuous and bounded.

(i) Find a bounded solution of (1), i.e., a solution satisfying

$$(2) \qquad \sup_{x\in\mathbb{R}^d,\, t>0} |u(x,t)| < \infty.$$

Is your solution unique among all solutions satisfying the condition (2)?

(ii) Find an explicit constant C that only depends on g so that your solution satisfies

$$|u(x,t)| \leq C$$

for all $(x,t) \in \mathbb{R}^d \times (0,\infty)$.

Problem 4.

Let $\Omega \subset \mathbb{R}^d$ be a bounded, smooth domain with outward unit normal n. Consider the initial-boundary value problem for the wave equation,

$$
(3) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial n}(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times [0, \infty), \\ u(x,0) = g(x), \quad \dfrac{\partial u}{\partial t}(x,0) = h(x) & \text{for } x \in \Omega. \end{cases}
$$

Show that (3) has at most one classical solution in $C^2([0,\infty) \times \overline{\Omega})$.

PART TWO

In the following we assume that the $d \times d$ matrix $(a_{ij}(x))$ is real and satisfies the uniform ellipticity condition,

$$
(4) \qquad \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2 \quad \text{for a.e. } x \in \mathbb{R}^d \text{ and for any } \xi \in \mathbb{R}^d,
$$

where $\mu > 0$. The domain Ω in $\mathbb{R}^d$ is assumed to be bounded, connected, and its boundary is smooth.

Problem 5.

Let $1 \leq p < \infty$. Show that $W_0^{1,p}(\mathbb{R}^d) = W^{1,p}(\mathbb{R}^d)$.

Problem 6.

(i) State the Lax-Milgram Theorem.

(ii) Consider

$$
\mathcal{L}(u) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u,
$$

where the matrix (a_{ij}) satisfies the ellipticity condition (4), and $a_{ij}, c \in L^\infty(\mathbb{R}^d)$. Show that there exists $\lambda > 0$, depending only on d, μ, $\|a_{ij}\|_{L^\infty(\mathbb{R}^d)}$, and Ω, such that for any $F \in H^{-1}(\Omega)$, the Dirichlet problem

$$
\begin{cases} \mathcal{L}(u) = F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}
$$

has a unique weak solution in $H_0^1(\Omega)$, provided

$$
c(x) \geq -\lambda \quad \text{for a.e. } x \in \Omega.
$$

Problem 7. Let
$$\mathcal{L} = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right),$$
where $a_{ij} \in L^\infty(\Omega)$ and satisfies the uniform ellipticity condition (4). Let $u \in H^1(\Omega)$ be a weak solution of $\mathcal{L}(u) = 0$ in Ω. Show that
$$\int_\Omega |\nabla u|^2 |\psi|^2 \, dx \le C \int_\Omega |u|^2 |\nabla \psi|^2 \, dx$$
for any $\psi \in C_0^1(\Omega)$, where C depends only on d, μ, and $\|a_{ij}\|_{L^\infty(\Omega)}$.

Problem 8. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of
$$-\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + c(x)u = 0 \quad \text{in } \Omega,$$
where $(a_{ij}(x))$ is a real symmetric matrix satisfying the ellipticity condition (4), and $a_{ij}, c \in C(\overline{\Omega})$. Assume $c(x) \ge 0$ in Ω. Show that
$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|.$$