Preliminary Examination in Partial Differential Equations

June 2020

Instructions

This is a three-hour examination. You are to work a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part**.

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

PART ONE

For the rest of the exam, the domain U in $\mathbb{R}^d$ is assumed to be bounded, connected, and smooth.

Problem 1. Suppose that $F \in C^2((0,\infty))$ and define u on $\mathbb{R}^d \backslash \{0\}$ by $u(x) = F(|x|)$. If we have $F''(r) + \frac{d-1}{r}F'(r) = 0$, show that $\Delta u = 0$.

Problem 2. Let $V \in C(\bar{U})$ be a nonnegative function. Consider the wave equation

$$\begin{cases} u_{tt} - \Delta u + V(x)u = f & (t,x) \in (0,\infty) \times U \\ u(t,x) = 0 & (t,x) \in [0,\infty) \times \partial U \\ u(0,x) = g(x), \quad u_t(0,x) = h(x) \end{cases} \tag{1}$$

Show that (1) has at most one classical solution.

Problem 3. Let $c > 0$ be a constant. Assume u is a classical solution of the equation

$$\begin{cases} u_t - \Delta u + cu = 0 \quad (t,x) \in (0,\infty) \times U \\ u = 0 \text{ on } \partial U \\ u(0,x) = g(x) \end{cases} \tag{2}$$

where $g \in C(\bar{U})$ is nonnegative. Show that there is a constant C that only depends on g so that

$$u(t,x) \le Ce^{ct}$$

Hint: Let $v(t,x) = e^{-ct}u(t,x)$. What equation does v satisfy?

Problem 4. Let $\Omega = B(0,1) \subset \mathbb{R}^2$, and $a, b \in C(\bar{\Omega})$. Suppose that $u \in C^1(\bar{\Omega})$ solves the equation

$$a(x,y)u_x + b(x,y)u_y = -u$$

in Ω, and that

$$xa(x,y) + yb(x,y) > 0 \tag{3}$$

if $(x,y) \in \partial\Omega$. Prove that $u = 0$.

Hint: Show that $\min_{\bar{\Omega}} u \ge 0$ and $\max_{\bar{\Omega}} u \le 0$. If the extremum is attained at a point on the boundary, use (3) to show that, at that point, $a(x,y)u_x + b(x,y)u_y$ and the normal derivative $\frac{\partial u}{\partial \nu}$ have the same sign.

PART TWO

Problem 5.

(1) Show that there is a constant C so that

$$\|u\|^2_{L^2(\mathbb{R}^3)} \le C\|Du\|_{L^2(\mathbb{R}^3)}\|u\|_{L^{6/5}(\mathbb{R}^3)}$$

for all $u \in C_c(\mathbb{R}^3)$.

(2) Use a scaling argument to show that the inequality

$$\|u\|^2_{L^2(\mathbb{R}^3)} \le C\|Du\|_{L^2(\mathbb{R}^3)}\|u\|_{L^{q}(\mathbb{R}^3)}$$

can only hold for all $u \in C_c(\mathbb{R}^3)$ when $q = 6/5$.

Problem 6.

Consider the equation

$$\begin{cases} -\Delta u + b(x)\frac{\partial u}{\partial x_1} = F \text{ in } U \\ u = 0 \text{ on } \partial U \end{cases} \tag{4}$$

where $b \in L^\infty(U)$ and $F \in H^{-1}(U)$.

(1) What does it mean for $u \in H^1_0(U)$ to be a weak solution of (4)?

(2) Show that if u is a weak solution of (4), then there is $C > 0$ so that

$$\|u\|_{H^1(U)} \le C(\|F\|_{H^{-1}(U)} + \|u\|_{L^2(U)})$$

Problem 7.

Prove the following special case of the Gagliardo-Nirenberg inequality: if $u \in C^1_c(\mathbb{R}^2)$, then $\|u\|_{L^2(\mathbb{R}^2)} \le C\,\|Du\|_{L^1(\mathbb{R}^2)}$

Problem 8. Suppose $u \in C^\infty(\bar{U}\times[0,\infty))$ is a solution of the heat equation $u_t - \Delta u = 0$ in $U \times (0,\infty)$ and that $u(x,t) = 0$ for $(x,t) \in \partial U \times [0,\infty)$.

Show that there is a constant $\lambda > 0$ so that

$$\int_U u(x,t)^2\,dx \le e^{-\lambda t}\int_U u(x,0)^2\,dt.$$

Hint: Let $\phi(t) = \int_U u(x,t)^2\,dt$ and differentiate ϕ. Recall that the Poincaré inequality is valid in U.