Preliminary Examination
in Partial Differential Equations

June 2022

Instructions

This is a three-hour examination. You are to work a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part**.

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy parts from two different problems. Indicate clearly what theorems and definitions you are using.

PART ONE

Problem 1. Let $B(0,r) = \{|x| < r\} \subset \mathbb{R}^n$. For any $u \in C^2(B(0,1)) \cap C(\overline{B(0,1)})$, define the function $\phi(r)$ by

$$\phi(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B(0,r)} u(y)\ d\sigma(y).$$

Here σ is the measure on the boundary $\partial B(0,r)$ of $B(0,r)$ and $\sigma(\partial B(0,1)) = \omega_{n-1}$.

(a.) Prove that

$$\phi'(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{B(0,r)} (\Delta u)(y)\ dy,$$

where $\Delta u := \sum_{j=1}^n u_{x_j x_j}$ is the Laplacian on $\mathbb{R}^n$.

(b.) If u is subharmonic on $B(0,1)$, that is, $(\Delta u)(x) \geq 0$, for all $x \in B(0,1)$, then show that

$$u(0) \leq \frac{1}{\omega_{n-1}} \int_{\partial B(0,1)} u(y)\ d\sigma(y).$$

Problem 2. Let $\Omega = \{|x| > 2022\}$. Assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a harmonic function so that $u(x) = 0$ when $|x| = 2022$ and $\lim_{|x|\to\infty} u(x) = 0$. Show that $u \equiv 0$.

Problem 3. Suppose $g \in L^\infty(\mathbb{R})$.

(a.) Prove that the function $u(x,t)$, for $x \in \mathbb{R}$ and $t > 0$ given by

$$u(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} g(y)\ dy,\ \ t > 0,$$

solves the heat equation:

$$u_t(x,t) = u_{xx}(x,t).$$

(b.) Prove that:

$$|u_x(x,t)| \leq \frac{C_0}{t^{1/2}} \|g\|_\infty,$$

where the constant $C_0 > 0$ is independent of g and $t > 0$.

Problem 4. Suppose that $\Omega \subset \mathbb{R}^3$ is an open, bounded, connected set with smooth boundary $\partial\Omega$. Let $u \in C^2(\overline{\Omega} \times [0,T))$ be a solution of the initial-boundary value problem for the wave equation:

$$u_{tt}(x,t) - \Delta u(x,t) = u_t(x,t),\ (x,t) \in \Omega \times [0,T),$$

with the initial-boundary condition:

$$u(x,t) = 0,\ (x,t) \in \partial\Omega \times [0,T).$$

We define the energy of u by

$$E(t) := \frac{1}{2} \int_\Omega \left\{ u_t(x,t)^2 + |Du(x,t)|^2 \right\}\ dx.$$

Prove the bounds:

$$0 \le E(t) \le e^{2t} E(0),\ t \in [0,T).$$

PART TWO

Problem 5.

Let u be in $C_c^\infty(\mathbb{R}^2)$. Prove that there exists a constant C that is independent of u so that

$$\left(\int_{\mathbb{R}^2} |u|^2\,dx\right)^{1/2} \le C \int_{\mathbb{R}^2} |Du|\,dx.$$

For the rest of the exam, the domain U in $\mathbb{R}^d$ is assumed to be an open, bounded, connected set with smooth boundary. We also assume that the $d\times d$ real matrix $\big(a^{ij}(x)\big)$ is symmetric, and satisfies the uniform ellipticity condition,

$$\mu|\xi|^2 \le \sum_{i,j=1}^d a^{ij}(x)\xi_i\xi_j \le \mu^{-1}|\xi|^2 \quad \textit{for a.e. } x\in\mathbb{R}^d \textit{ and for any } \xi\in\mathbb{R}^d, \tag{1}$$

where $\mu > 0$.

Problem 6.

Consider the equation

$$\begin{cases} -\sum_{i,j=1}^d \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{j=1}^d b^j(x)u_{x_j} = F \text{ in } U \\ u = 0 \text{ on } \partial U \end{cases} \tag{2}$$

where $b^j \in L^\infty(U)$ and $F\in L^2(U)$.

(a.) What does it mean for $u\in H_0^1(U)$ to be a weak solution of (2)?

(b.) Show that there is a constant $c>0$ so that, if $\|b^j\|_{L^\infty(U)} \le c$ for all j, then (2) has a unique weak solution.

Problem 7.

Let (a^{ij}) be a $d\times d$ matrix as above. Assume, moreover, that $a^{ij}\in C^1(\overline{U})$. Let $u\in C^2(U)$ be a (classical) solution of

$$\sum_{i,j=1}^d \left(a^{ij}(x)u_{x_i}\right)_{x_j} = 0$$

Show that, if $y\in U$ so that the closed ball $\overline{B(y,2)}\subset U$, then we have a positive constant C so that

$$\int_{B(y,1)} \sum_{|\alpha|=2} |D^\alpha u|^2\,dx \le C \int_{B(y,2)} |Du|^2\,dx.$$

Problem 8.

Let U be a domain as above. Consider the Rayleigh quotient

$$R[\phi] = \frac{\displaystyle\int_U |D\phi|^2 \, dx}{\displaystyle\int_U |\phi|^2 \, dx}, \quad \phi \in H_0^1(U) \setminus \{0\}.$$

(a.) Let $\lambda \in \mathbb{R}$. Give a careful definition of what it means for $u \in H_0^1(U)$ to be a weak solution of the equation

$$-\Delta u = \lambda u, \qquad \text{in } U.$$

(b.) Suppose $\lambda = R[u]$ is the minimum value of the Rayleigh quotient on $H_0^1(U) \setminus \{0\}$ and this minimum value occurs for some $u \in H_0^1(U) \setminus \{0\}$. Prove that u is a weak solution of the eigenvalue equation $-\Delta u = \lambda u$.

Hint: For any arbitrary $v \in H_0^1(U)$, consider the function $\psi(t) := R[u+tv]$, and compute $\psi'(0)$.