Preliminary Examination
in Partial Differential Equations

May 2024

Instructions

This is a three-hour examination. You are to work a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part**.

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

NOTATION: For $r > 0$ and $x \in \mathbb{R}^d$, we let $B(x,r) := \{y \in \mathbb{R}^d \mid |x-y| < r\}$, the open ball of radius r centered at $x \in \mathbb{R}^d$. The symbol D_j denotes the partial derivative $D_j := \dfrac{\partial}{\partial x_j}$, and the symbol D denotes the gradient. For a function $u(x,t)$, we write $u_t := \dfrac{\partial u}{\partial t}$ and $u_{x_j} := \dfrac{\partial u}{\partial x_j}$. The Laplacian in Cartesian coordinates is $\Delta := \sum_{j=1}^d \dfrac{\partial^2}{\partial x_j^2}$. Any domain U in $\mathbb{R}^d$ is assumed to be an open, bounded, connected set with smooth boundary. The k-times differentiable functions of compact support on an open set $U \subset \mathbb{R}^d$ are denoted by $C_c^k(U)$.

PART ONE

Problem 1. Assume that $u \in C^2(B(0,1)) \cap C(\overline{B(0,1)})$, and define the function $\phi(r)$ by

$$\phi(r) := \frac{1}{\omega_{d-1} r^{d-1}} \int_{\partial B(0,r)} u(y)\ d\sigma(y).$$

Here σ is the measure on the boundary $\partial B(0,r)$ of $B(0,r)$ and $\sigma(\partial B(0,1)) = \omega_{d-1}$.

(a.) Prove that

$$\phi'(r) = \frac{1}{\omega_{d-1} r^{d-1}} \int_{B(0,r)} (\Delta u)(y)\ dy.$$

(b.) If u is subharmonic on $B(0,1)$, that is, $(\Delta u)(x) \geq 0$, for all $x \in B(0,1)$, then show that

$$u(0) \leq \frac{1}{\omega_{d-1}} \int_{\partial B(0,1)} u(y)\ d\sigma(y).$$

Problem 2. Show that the equation

$$\begin{cases} u_t - \Delta u + \sum_{j=1}^d u_{x_j} = f & (t,x) \in (0,\infty) \times U \\ u(t,x) = 0 & (t,x) \in [0,\infty) \times \partial U \\ u(0,x) = g(x) \end{cases}$$

has at most one classical solution.

Problem 3. Suppose $g \in L^\infty(\mathbb{R})$.

(a.) Prove that the function $u(x,t)$, for $x \in \mathbb{R}$ and $t > 0$ given by

$$u(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} g(y)\ dy,\ t > 0,$$

solves the heat equation:

$$\partial_t u(x,t) = \partial_x^2 u(x,t).$$

(b.) Prove that:

$$\left| \frac{\partial u}{\partial x}(x,t) \right| \leq \frac{C_0}{t^{1/2}} \|g\|_\infty,$$

where the constant $C_0 > 0$ is independent of g and $t > 0$.

Problem 4. Let $u \in C^2(\overline{U} \times [0,T))$ be a solution of the initial value problem for the wave equation:

$$u_{tt}(x,t) - \Delta u(x,t) = u_t(x,t),\ (x,t) \in U \times (0,T),$$

with boundary condition:

$$u(x,t) = 0, \ (x,t) \in \partial U \times [0,T).$$

We define the energy of u by

$$E(t) := \frac{1}{2} \int_U \left\{ u_t(x,t)^2 + |Du(x,t)|^2 \right\} \, dx.$$

Prove the bounds:

$$0 \le E(t) \le e^{2t} E(0), \ t \in [0,T).$$

PART TWO

For the rest of the exam, we also assume that the $d \times d$ real matrix $\big(a^{ij}(x)\big)$ is symmetric, and satisfies the uniform ellipticity condition,

$$\mu|\xi|^2 \leq \sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \leq \mu^{-1}|\xi|^2 \quad \textit{for a.e. } x \in \mathbb{R}^d \textit{ and for any } \xi \in \mathbb{R}^d, \tag{1}$$

where $\mu > 0$.

Problem 5.

Assume $F : \mathbb{R} \to \mathbb{R}$ is C^1, with F' bounded. Let $u \in W^{1,p}(U)$ for some $1 < p < \infty$, and $w = F(u)$. Show that

$$\frac{\partial w}{\partial x_i} = F'(u)\frac{\partial u}{\partial x_i} \quad \text{and} \quad w \in W^{1,p}(U).$$

Problem 6.

Let $U = \{1/2 < |x| < 1\}$, and consider the equation

$$\begin{cases} -\sum_{i=1}^{d}\left(|x|u_{x_i}\right)_{x_i} = F \text{ in } U \\ u = 0 \text{ on } \partial U \end{cases} \tag{2}$$

where $F \in L^2(U)$.

(a.) What does it mean for $u \in H_0^1(U)$ to be a weak solution of (2)?
(b.) Show that (2) has a unique weak solution.
(c.) Does anything change in your argument if $U = \{|x| < 1\}$?

Problem 7.

Let

$$\mathcal{L} = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$$

where a^{ij} satisfy (1). Suppose $u \in H^1(U)$ is a weak solution of $\mathcal{L}(u) = 0$ in U. Show that

$$\int_U |Du|^2|\psi|^2 \leq C\int_U |u|^2|D\psi|^2$$

for any $\psi \in C_0^\infty(U)$, where C depends only on d and μ defined in (1).

Problem 8.

Assume that a^{ij} are continuous and satisfy (1), and $u \in C^2(U)$ satisfies

$$-\sum_{i,j=1}^{d} a^{ij}(x) u_{x_i x_j} \geq 0, \quad \forall x \in U.$$

Prove that either u is a constant or

$$u(x) > \inf_U u$$

for all $x \in U$. Note that u may not be in $C(\overline{U})$.

Hint: Show that the set

$$V = \{x \in U : u(x) > \inf_U u\}.$$

is both open and closed.