Preliminary Examination
in Partial Differential Equations

June 2025

Instructions

This is a three-hour examination. You are to work a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part**. Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems.

Indicate clearly what theorems and definitions you are using.

PART ONE

Problem 1.

(a) State the mean value formula for harmonic functions, and its converse.

(b) Let Ω be a smooth, bounded, connected domain. Assume that $u_n : \Omega \to \mathbb{R}$ is a sequence of harmonic functions, and that u_n converges uniformly to a function $u \in C^2(\Omega)$. Show that u is also harmonic.

Problem 2. Let $\Omega = \{|x| > 1\} \subset \mathbb{R}^d$, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a harmonic function in Ω. Show that, if $u = 0$ on $\partial\Omega$, and $\lim_{|x|\to\infty} |u(x)| = 0$, then $u = 0$ in Ω. Hint: Use the maximum principle.

Problem 3. Suppose $g \in L^\infty(\mathbb{R}^d)$.

(a) Prove that the function $u(x,t)$, for $x \in \mathbb{R}^d$ and $t > 0$, given by

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\frac{-|x-y|^2}{4t}} g(y)\, dy,$$

solves the heat equation:

$$u_t(x,t) = \Delta u(x,t), \text{ for } x \in \mathbb{R}^d \text{ and } t > 0.$$

(b) Prove that the solution $u(x,t)$ satisfies the initial condition:

$$\lim_{t\to 0} u(x,t) = g(x).$$

Problem 4. Suppose that $\Omega \subset \mathbb{R}^3$ is an open, bounded, connected set with smooth boundary $\partial\Omega$. Let $u \in C^2(\overline{\Omega} \times [0,T))$ be a solution of the initial-boundary value problem for the wave equation:

$$u_{tt}(x,t) - \Delta u(x,t) = u_t(x,t),\ (x,t) \in \Omega \times [0,T),$$

with the initial-boundary condition: $u(x,t) = 0, (x,t) \in \partial\Omega \times [0,T)$. We define the energy of u by

$$E(t) := \frac{1}{2} \int_\Omega [\ |u_t(x,t)|^2 + |\nabla u(x,t)|^2]\ dx.$$

Prove the upper bound:

$$0 \le E(t) \le e^{2t} E(0),\ t \in [0,T)$$

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PART TWO

In the following we assume that the $d \times d$ matrix $\big(a_{ij}(x)\big)$ is real and satisfies the uniform ellipticity condition : $a_{ij} \in L^\infty(\mathbb{R}^d)$ and

$$(1) \qquad \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2 \quad \text{for a.e. } x \in \mathbb{R}^d \text{ and for any } \xi \in \mathbb{R}^d,$$

where $\mu > 0$. The domain Ω in $\mathbb{R}^d$ is assumed to be bounded, connected, and its boundary is smooth.

Problem 5.

(a) Let $u, v \in L^1_{loc}(\Omega)$. What does it mean if v is said to be the weak derivative $\frac{\partial u}{\partial x_i}$ of u? State the definition.

(b) Let $u \in L^1_{loc}(\Omega)$. Suppose the weak derivatives $\frac{\partial u}{\partial x_i} = 0$ in Ω for $1 \leq i \leq d$. Show that u is constant in Ω. Hint: Consider $u_\varepsilon = u * \eta_\varepsilon$, where $\{\eta_\varepsilon : \varepsilon > 0\}$ is an approximation to identity, and compute $\frac{\partial u_\varepsilon}{\partial x_i}$.

Problem 6.

(a) Let $f \in L^2(\Omega)$. What does it mean if $u \in H^1(\Omega)$ is said to be a weak solution of

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left\{a_{ij}(x)\frac{\partial u}{\partial x_j}\right\} = f$$

in Ω?

(b) Let $u \in H^1(\Omega)$ be the weak solution in part (a). Show that

$$\|\nabla u\|_{L^2(V)} \leq C\left\{\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right\},$$

where $V \subset\subset \Omega$ and C depends on V, Ω, and $\{a_{ij}\}$.

Problem 7. Consider the nondivergence form operator,

$$\mathcal{L}(u) = -\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j},$$

where the matrix (a_{ij}) is smooth in $\mathbb{R}^d$ and satisfies the ellipticity condition (1). Let u be a smooth solution of $\mathcal{L}(u) = 0$ in Ω.

(a) Let

$$v = |\nabla u|^2 + \lambda u^2.$$

Show that $\mathcal{L}(v) \leq 0$ in Ω, if $\lambda \geq \lambda_0$ for some large λ_0 depending only on d, Ω and (a_{ij}).

(b) Show that

$$\|\nabla u\|_{L^\infty(\Omega)} \le C\Big\{ \|\nabla u\|_{L^\infty(\partial\Omega)} + \|u\|_{L^\infty(\partial\Omega)} \Big\},$$

where C depends only on d, Ω, and (a_{ij}).

Problem 8. Recall that $H_0^2(\Omega) = W_0^{2,2}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^{2,2}(\Omega)$. Show that for any $u \in H_0^2(\Omega)$,

$$\int_\Omega |\Delta u|^2\, dx = \sum_{i,j=1}^d \int_\Omega \Big| \frac{\partial^2 u}{\partial x_i \partial x_j} \Big|^2\, dx.$$