1 Let $f_1 : X \longrightarrow X_1$ and $f_2 : X \longrightarrow X_2$ be continuous functions between topological spaces for which f_2 is an *imbedding*, i.e., a homeomorphism onto its image. Define $f : X \longrightarrow X_1 \times X_2$ by $f(x) = (f_1(x), f_2(x))$. Prove that f is also an imbedding.

2 Let X and Y be two metric spaces with distance functions d_X and d_Y respectively, and let $f : X \longrightarrow Y$ be any map. We'll say that f has a finite limit $a \in Y$ at infinity, if for every $\varepsilon > 0$, there is a compact set $K \subset X$ such that $d_Y(f(x), a) < \varepsilon$ for all $x \in X - K$. Prove that if f is continuous and has a finite limit at infinity, then f is uniformly continuous.

3 A metric space X is *ultrametric*, if its distance function d satisfies

$$d(x, z) \leq \max\{d(x, y), d(y, z)\},$$

for all $x, y, z \in X$.

Let X be an ultrametric space. Prove that:

(a) Every two open balls of the same radius in X are either disjoint or equal.
(b) If X has more than one point, then X is disconnected.

4 Let $A \subset U \subset X$, where A is closed, U is open, and X is normal. Prove that there exists a homeomorphism $h : X \times \mathbb{R} \longrightarrow X \times \mathbb{R}$ for which $h(a, 0) = (a, 1)$, for all $a \in A$, and $h(x, t) = (x, t)$, for all $(x, t) \in (X - U) \times \mathbb{R}$.

5 Let $X = \Pi_{n=1}^{\infty} X_n$ have the product topology. Prove that if $X \neq \varnothing$ and locally compact, then (1) every X_n is locally compact and (2) there is an N for which X_n is compact, for all $n \geq N$.

6 Let $I = [0, 1]$ and $X = C(I, I)$ be the space of continuous functions $f : I \longrightarrow I$ with the topology induced by the uniform metric

$$d(f, g) = \sup_{t \in I} |f(t) - g(t)|.$$

(a) Is X connected?
(b) Is X compact?

7 Let $X \xrightarrow{f} Y \xrightarrow{g} X$ be continuous functions between topological spaces for which $fg \simeq \mathrm{id}_Y$. For any $y_0 \in Y$, prove that

$$g_* : \pi_1(Y, y_0) \longrightarrow \pi_1(X, g(y_0))$$

is one-to-one.

8 Prove that every continuous map $S^2 \longrightarrow T^2$ from the 2-sphere S^2 to the 2-torus $T^2 = S^1 \times S^1$ is inessential.