1 Let $f : [0,1] \longrightarrow [0,1]$ be a continuous function for which $\frac{1}{2} \notin f([0,1])$. Prove that there is an $x \in [0,1]$ for which $|f(x) - x| > \frac{1}{2}$.

2 Let $I = [0,1]$ and $\mathbb{R}$ have the usual topology, and let

$$\pi_1 : I \times \mathbb{R} \longrightarrow I$$

and

$$\pi_2 : I \times \mathbb{R} \longrightarrow \mathbb{R}$$

be the two projection maps. Prove that π_2 is closed but π_1 is not.

3 Prove that a locally compact Hausdorff space is regular.

4 Let A and B be two closed subsets of $\mathbb{R}^n$. Consider

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

(a) Prove that if A is bounded, then $A + B$ is closed.
(b) Give an example showing that if A and B are unbounded, then $A + B$ need not be closed.

5 Define $p : [0,1] \longrightarrow S^1$ by $p(t) = (\cos 2\pi t, \sin 2\pi t)$. Prove that if $f : [0,1] \longrightarrow X$ is continuous and $f(0) = f(1)$, then there is a unique continuous function $\tilde{f} : S^1 \longrightarrow X$ for which $\tilde{f} \circ p = f$.

6 Let X be a compact metric space and let $\mathcal{C}(X,X)$ be given the usual sup metric. Prove that if (f_n), (g_n) are convergent sequences in $\mathcal{C}(X,X)$ and $\lim f_n = f$, $\lim g_n = g$, then $\lim g_n \circ f_n = g \circ f$.

7 Let Y be a compact subset of $\mathbb{R}^n$ which is a retract of some open subset U of $\mathbb{R}^n$. Prove that there is an $\varepsilon > 0$ which satisfies the following property: If X is any space and $f, g : X \longrightarrow Y$ are continuous functions which satisfy $d(f(x), g(x)) \leq \varepsilon$, for all $x \in X$, then f is homotopic to g. [Y is a retract of U if $Y \subset U$ and there is a continuous function $r : U \longrightarrow Y$ such that $r(y) = y$ for all $y \in Y$.]

8 Let P^2 be the projective plane. Prove that every covering map $p : P^2 \longrightarrow P^2$ is a homeomorphism.