1 Let $X_1 \subset X_2 \subset \cdots$ be a sequence of topological spaces such that each X_n is a closed subspace of X_{n+1}, and let

$$X = \bigcup_{n=1}^{\infty} X_n.$$

Define

$$\mathcal{T} = \{U \mid U \subset X \text{ and each } U \cap X_n \text{ is open in } X_n\}.$$

(a) Prove that $(X, \mathcal{T})$ is a topological space.
(b) Prove that each X_n is closed in $(X, \mathcal{T})$.

2 (a) Show that every path connected space is connected, but that connected spaces need not be path connected.
(b) Give an argument that connected open subsets of the plane are path connected. What topological properties are used in the argument?

3 (a) Given that X and Y are finite spaces, each of which contains a 1–1 continuous image of the other, prove that X and Y are homeomorphic.
(b) Show that if the word 'finite' in the above statement is replaced with 'connected', the statement is false.

4 Let X be a topological space, (Y, d) a metric space. Let $\mathcal{C}(X, Y)$ be the set of all continuous functions $f : X \longrightarrow Y$. For $f \in \mathcal{C}(X, Y)$ and a positive continuous function $\delta : X \longrightarrow \mathbb{R}_+$, define

$$B(f, \delta) = \{g \in \mathcal{C}(X, Y) \mid d(f(x), g(x)) < \delta(x) \text{ for all } x \in X\}.$$

It is easy to prove that the sets $B(f, \delta)$ form a basis for a topology on $\mathcal{C}(X, Y)$ called the *fine topology.*

Consider the case $X = Y = \mathbb{R}$. Let $A \subset \mathcal{C}(\mathbb{R}, \mathbb{R})$ be the subset of all strictly positive functions.

(a) Show that the constant function 0 belongs to the closure of A in the fine topology.
(b) Show that there is no sequence of strictly positive functions converging to 0 in the fine topology.
(c) Conclude that the fine topology on $\mathcal{C}(\mathbb{R}, \mathbb{R})$ is not metrizable.

5 Let $K \subset \mathbb{R}^n$ be a compact space and $f : K \longrightarrow \mathbb{R}_+$ be a continuous function. Prove that the "f-neighborhood"

$$N_f(K) = \{x \in \mathbb{R}^n \mid \text{for some } k \in K,\ \|x - k\| \leq f(k)\}$$

of K is also compact.

6 Classify the following spaces:

(a) The 1-point compactification of the discrete integers.
(b) The 1-point compactification of the open unit interval $(0, 1)$.
(c) The 1-point compactification of the half-open unit interval $[0, 1)$.
(d) The convergent sequence $\{0, 1, 1/2.1/3, \ldots, 1/n, \ldots\}$.
(e) The quotient space of the reals obtained from the equivalence relation: xRy if and only if $x - y$ is an integer.
(f) The rationals in $[0, 1)$.
(g) The rationals in $(0, 1)$.
(h) The closed unit interval $[0, 1]$.

Note: Group the spaces into homeomorphism classes, giving as much reasoning as time permits. Especially, give an argument that (f) and (g) are homeomorphic.

7 Let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the standard unit circle. Prove that every homeomorphism $h : S^1 \longrightarrow S^1$ is homotopic to either the identity map or the reflection in the x-axis.

8 Let $p : E \longrightarrow B$ be a covering map with E path connected. Prove that every continuous map $F : E \longrightarrow E$ satisfying $p \circ F = p$ is surjective.