1 Let X be a topological space. For $A \subset X$, we write $\mathrm{Bd}(A)$ to designate the boundary of A.

(a) Prove that if $U \subset X$ is open, then $\mathrm{Bd}(\mathrm{Bd}(U)) = \mathrm{Bd}(U)$.
(b) Prove that if $A \subset X$ and $\mathrm{Bd}(A) = A$, then there is an open set $U \subset X$ such that $A = \mathrm{Bd}(U)$.

2 Let X be Hausdorff and let A be a locally compact subspace. Prove that $A \subset U \subset X$, where U is open and A is closed in U.

3 Prove that if a space has a finite number of connected components, then these components are open.

4 You are given a compact space X and a metric space Y. Let the space $\mathcal{C}(X, Y)$ of continuous functions have the sup metric. Prove that if $C \subset Y$ is closed and $U \subset X$ is open, then the set

$$S = \left\{ f \in \mathcal{C}(X, Y) \mid f^{-1}(C) \subset U \right\}$$

is open in $\mathcal{C}(X, Y)$.

5 Let Y be the subspace of $\mathbb{R}^2$ defined by

$$Y = \{(x, y) \mid xy = 0\}.$$

Prove that every continuous map $f : A \longrightarrow Y$ from a closed subset A of a normal space X can be extended to a continuous map $F : X \longrightarrow Y$.

6 A map $f : X \longrightarrow Y$ between metric spaces is *Cauchy-preserving* if for every Cauchy sequence (x_n) in X, the sequence $(f(x_n))$ is a Cauchy sequence in Y.

(a) Prove that if f is uniformly continuous, then f is Cauchy-preserving.
(b) Is the converse of (a) true?

7 Let X be a topological space, $f : [0, 1] \longrightarrow X$ a continuous map and $\gamma : [0, 1] \longrightarrow X$ any continuous path satisfying $f(0) = \gamma(0)$. Prove that there is a homotopy $F : [0, 1] \times [0, 1] \longrightarrow X$ of f satisfying $F(0, t) = \gamma(t)$ for all $t \in [0, 1]$.

8 Let $p : E \longrightarrow B$ be a covering map and let $f : X \longrightarrow B$ be a continuous function, where X is connected. Prove that any two liftings of f which agree at a point $x_0 \in X$ must be equal.