Topology Preliminary Examination
January 7, 2005

1. Prove: X is Hausdorff if and only if the diagonal $\Delta_X = \{(x,x) | x \in X\}$ is closed in the product space $X \times X$.

2. Prove that in a metrizable space, every closed set is the intersection of a countable family of open sets, and every open set is the union of a countable family of closed sets.

3. Let X be a topological space satisfying the following axiom: If A and B are any two disjoint closed subsets of X, then there exist two disjoint open sets U, V such that $A \subset U$ and $B \subset V$. Show that the relation R: $\overline{\{x\}} \cap \overline{\{y\}} \neq \emptyset$ between two points x, y of X is an equivalence relation.

4. Show that $\mathbb{R}^\omega$ is not locally compact in the box topology.

5. Let X be a connected space and let $X = A \cup B$, where A and B are closed subsets. Prove that if $A \cap B$ is connected, then A and B are also connected.

6. Let X be a metric space which is totally bounded, i.e. for every $\epsilon > 0$, there is a finite covering of X by ϵ-balls. Prove that X has a countable dense subset.

7. (a) Prove that if X is contractible, then $\pi_1(X, x_0)$ is trivial for all $x_0 \in X$.

(b) Let P^2 be the projective plane, i.e. the quotient space obtained from the unit disk B^2 by identifying each x on the unit circle with its antipodal point $-x$. Show that P^2 is not contractible.

8. Let $O \in U \subset B^2 \subset \mathbb{R}^2$, where O is the origin and U is open in $\mathbb{R}^2$. Prove that there does not exist a retraction $r : \overline{U} \to Bd(U)$.