Preliminary Exam – Topology – January 4, 2006

1. Prove that if X is a space which is sequentially compact, then X must be limit point compact.

2. Let X be a locally connected space and let C be the collection of all its components.

   (a) Give an example in which C is uncountable.

   (b) Prove that if X is additionally assumed to be separable, then C is countable.

3. Let $p : \mathbb{R}^1 \to S^1$ be the covering map defined by $p(x) = (\cos(2\pi x), \sin(2\pi x))$ and let $f : [0,1] \to S^1$ be a path. Assume that f_1 and f_2 are two liftings of f for which $f_1(0) = f_2(0) + 1$. Prove that $f_1(t) = f_2(t) + 1$ for all $t \in [0,1]$.

4. Show that a continuous map from a compact space onto a Hausdorff space is a quotient map.

5. Compute the fundamental group of the projective plane.

6. Show that the comb space is path connected but not locally path connected. (By the comb space we mean the subspace C of the Euclidean plane $\mathbb{R}^2$ given by $C = ([0,1] \times 0) \cup (A \times [0,1])$, where $A = \{x | x = 0 \text{ or } x = \frac{1}{n} \text{ for } n \text{ a positive integer }\}$).

7. Construct a distance on the real line with respect to which the real line is not complete.

8. A topological property $\mathcal{P}$ is **weakly hereditary** if every closed subspace of a space with property $\mathcal{P}$ has property $\mathcal{P}$. Which of the following properties is weakly hereditary? Prove or give a counterexample: *connected, normal, locally compact Hausdorff, Lindelöf.*