# Topology Preliminary Exam

*January 4, 2007*

**Note:** $\mathbb{R}$ denotes the reals with the usual topology. $\mathbb{R}^2$ denotes the product space $\mathbb{R} \times \mathbb{R}$.

1. Two subspaces $A$ and $B$ of $\mathbb{R}$ are *similar* if there is a homeomorphism $f$ of $\mathbb{R}$ such that $f(A) = B$.

   i) True or false and why? Any subspace $A$ of $\mathbb{R}$ which is homeomorphic with $[0,1]$ is similar to $[0,1]$.

   ii) Find two connected subspaces $A$ and $B$ of $\mathbb{R}$ which are homeomorphic but not similar.

2. Prove that any space with the fixed point property must be $T_0$. (Recall: $X$ has the fixed point property if for each continuous function $f: X \to X$, there is a point $x \in X$ such that $f(x) = x$; $X$ is $T_0$ if for any two distinct points of $X$ there is an open set containing one but not the other.)

3. A space $X$ is said to be *weakly locally path connected (WLPC)* if for each $x \in X$ and open set $U$ containing $x$ there exists an open set $V$ such that $x \in V \subset U$ and such that any two points of $V$ can be connected by a path in $U$. Prove that a retract of a WLPC space must be WLPC.

4. Let $M$ be the space obtained from a torus by collapsing a meridian circle to a point and $L$ the space obtained from a torus by collapsing a longitudinal circle to a point. Show that the two quotient spaces $M$ and $L$ are homeomorphic.

5. Show that a normal space $X$ is separable if and only if there is a countable subspace $A \subset X$ such that the restriction map $\rho: \mathcal{C}(X, \mathbb{R}) \to \mathcal{C}(A, \mathbb{R})$ is injective. [$\mathcal{C}(X, \mathbb{R})$ denotes the set of all continuous real valued functions on $X$ and $\rho(f)$ equals $f$ restricted to $A$.]

6. Let $f: \mathbb{R}^2 \to X$ be a function which satisfies the following property.
   For each continuous function $a: [0,1] \to \mathbb{R}^2$, the composition $f \circ a: [0,1] \to X$ is continuous.
   Prove that $f$ must be continuous.

7. Let $X = \mathbb{R}^2 - \{(0,0),(1,0)\}$ and let $Y = \mathbb{R}^2 - \{(0,0)\}$. Prove that there is no covering map $p: X \to Y$.

8. Prove that a topological space which admits a universal covering space must be semilocally simply connected.