DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
January 3, 2008

1 Let $[0,1]^\omega$ be the countably infinite product of $[0,1]$ with itself, i.e.

$$[0,1]^\omega = \prod_{n=1}^{\infty} X_n,$$

where $X_n = [0,1]$ for all $n \in \mathbf{Z}_+$. For $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n)$, define

$$d(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} |x_n - y_n| \cdot 2^{-n}.$$

(a) Show that d is a metric on $[0,1]^\omega$.
(b) Show that the topology defined by the metric d is the product topology.

2 Let X be a Hausdorff space in which each point has a compact neighborhood, i.e. belongs to an open set contained in a compact subspace.

(a) Show that this property is inherited by each open subspace $U \subset X$.
(b) Is (a) still true if "compact" is replaced by "connected"? Provide a proof or provide a counterexample.

3 Let the real plane $\mathbf{R}^2$ be partitioned into vertical lines $\{b\} \times \mathbf{R}$ outside the strip $(-1,1) \times \mathbf{R}$ and by the curves

$$y = \frac{1}{1-x^2} + a$$

parameterized by real numbers a inside the strip.

(a) Is the quotient space with respect to this partition Hausdorff? Justify your answer.
(b) Same question if the curves are

$$y = \frac{x}{1-x^2} + a.$$

4 Let X be a metric space. Prove that X is compact if and only if every continuous function $f : X \longrightarrow \mathbf{R}$ is bounded. [You may find Tietze's Extension Theorem useful.]

5 For two spaces X, Y, let $\mathcal{C}(X, Y)$ be the set of continuous functions $X \longrightarrow Y$ equipped with the compact-open topology. Let X, Y, Z be three spaces and let $g : Y \longrightarrow Z$ be a continuous function. Define a function

$$\varphi : \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X, Z)$$

by $\varphi(f) = g \circ f$. Prove that φ is continuous.

6 Let $m, n \geq 1$, $A \subset \mathbf{R}^n$ be a bounded (in Euclidean metric) subspace, and $f : A \longrightarrow \mathbf{R}^m$ be a homeomorphism. Prove that f is not uniformly continuous.

7 Let $A, B \subset \mathbf{R}^2$ be the subsets defined by

$$A = (\mathbf{R} \times \{0\}) \cup \{(0,1), (0,-1)\}, \quad B = (\mathbf{R} \times \{0\}) \cup \{(0,1), (0,2)\}.$$

Is there a homeomorphism $h : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ such that $h(A) = B$? Justify your answer.

8 Let $p : E \longrightarrow B$ be a covering map with E path connected. Suppose p has a section, i.e. there is a continuous map $s : B \longrightarrow E$ such that $p \circ s = \mathrm{id}_B$. Prove that p is a homeomorphism.