DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
January 8, 2009

1 If X is a metric space and $Y \subset X$ is complete, prove that Y is closed.

2 Define $A = \{(x,y) \in \mathbf{R}^2 \mid x > 0 \text{ and } y < x^2\}$ and, for any $\alpha > 0$, define $f_\alpha : \mathbf{R} \longrightarrow \mathbf{R}$ by

$$f_\alpha(x) = \begin{cases} 0, & \text{if } (x, \alpha x) \in A, \\ 1, & \text{if } (x, \alpha x) \notin A. \end{cases}$$

Determine the values of x for which f_α is continuous.

3 Let A be a subset of the Euclidean space $\mathbf{R}^n$. Prove that if $\mathrm{Bd}(A)$ is connected, then $\mathrm{Cl}(A)$ is also connected. Here $\mathrm{Bd}(A) = \mathrm{Cl}(A) - \mathrm{Int}(A)$ is the difference between the closure and the interior of A in $\mathbf{R}^n$. [*Hint:* $\mathbf{R}^n$ is connected, even path connected.]

4 Find a metric space and two balls in it such that the ball with the smaller radius contains the ball with the larger radius and does not coincide with it.

5 Let $F : X \times I \longrightarrow Y$ be a homotopy. Prove that the map $\gamma : I \longrightarrow \mathcal{C}(X,Y)$ defined by $\gamma(t)(x) = F(x,t)$ is continuous. Here $I = [0,1]$ and $\mathcal{C}(X,Y)$ is the space of continuous functions $X \longrightarrow Y$ with the compact-open topology.

6 Define an equivalence relation $\sim$ on $\mathbf{R}$ by

$$x \sim y \quad \Leftrightarrow \quad x = y \quad \text{or} \quad x, y \in [0,1].$$

Prove that the quotient space $\mathbf{R}/{\sim}$ is homeomorphic to $\mathbf{R}$.

7 Prove that every covering map $p : X \longrightarrow B$ with simply connected B and path connected X is a homeomorphism.

8 Let $A = S^1 \times \{(0,0)\} \subset \mathbf{R}^2 \times \mathbf{R}^2 = \mathbf{R}^4$, where S^1 is the unit circle in $\mathbf{R}^2$. Prove that $\mathbf{R}^4 - A$ is simply connected.