DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
January 7, 2010

1 Let $M_3(\mathbb{R})$ denote the space of 3×3 matrices with real entries. Observe that $M_3(\mathbb{R})$ naturally has the structure of a metric space when we identify $M_3(\mathbb{R})$ with $\mathbb{R}^9$ (there are 9 real entries in a 3×3 matrix). Recall that the *trace* function

$$\mathrm{tr} : M_3(\mathbb{R}) \to \mathbb{R}$$

is defined for every $A \in M_3(\mathbb{R})$ by

$$\mathrm{tr}(A) = A_{1,1} + A_{2,2} + A_{3,3}.$$

Set

$$Z = \{A \in M_3(\mathbb{R}) \mid \mathrm{tr}(A) = 0\},$$

and endow Z with the subspace topology. Prove or disprove the following statements:

(a) Z is a closed subspace of $M_3(\mathbb{R})$.
(b) Z is compact.
(c) Z is paracompact.

2 Let X be compact and let $f_n : X \longrightarrow \mathbb{R}$ ($n \geq 1$) and $g : X \longrightarrow \mathbb{R}$ be continuous functions such that (f_n) converges pointwise to g.

(a) Give an example showing that the convergence need not be uniform.
(b) Prove that if $f_n(x) \geq f_{n+1}(x)$ for all $x \in X$ and $n \geq 1$, then the convergence is uniform.

3 Let X be locally connected and let U be an open subset of X. Prove that if C is a component of U, then $\mathrm{Bd}(C) \subset X - U$.

4 Let a, b be two real numbers, let

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid |ax + by| \leq 1\right\},$$

and let $f : A \longrightarrow \mathbb{R}$ be the map given by $f(x, y) = x$. For which pairs a, b is f a closed map?

5 For any X let $\mathcal{C}([0,1], X)$ be the space of all paths in X with the compact-open topology. Define $\alpha : \mathcal{C}([0,1], X) \longrightarrow \mathcal{C}([0,1], X)$ by $\alpha(f)(t) = f(1-t)$. Prove that α is a homeomorphism.

6 Consider the function $f : \mathbb{R} \to \mathbb{R}^2$ given by

$$f(t) = (t^2 - 1, t(t^2 - 1)).$$

What is the fundamental group of the range $\mathrm{Im}(f)$ of f? Provide justification for your answer.

7 Prove that every two continuous maps $S^2 \longrightarrow \mathbb{R}^2 - \{(0,0)\}$ are homotopic.

8 Prove that the 2-sphere S^2 is not homeomorphic to any subspace of $\mathbb{R}^2$.