DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
January 6, 2011

1 Let $\mathbf{R}$ have the finite complement topology. Is the subspace topology on $\mathbf{Z} \subset \mathbf{R}$ the same as the discrete topology? Justify your answer.

2 Let $X = [-1, 1]$ and define a topology $\mathcal{T}$ on X by declaring a set U to be open if and only if either $0 \notin U$ or $(-1, 1) \subset U$.

(a) Is the set $\{1\}$ open, closed, neither, or both?
(b) Is the set $\{-1, 1\}$ open, closed, neither, or both?
(c) Is the set $(0, 1)$ open, closed, neither, or both?
(d) Is this the same as the discrete topology on $[-1, 1]$?
(e) What is the subspace topology on $X - \{0\}$?
(f) Is X compact?
(g) Is X connected? locally connected?
(h) Is X Hausdorff?

3 Let X and Y be topological spaces and $f : X \longrightarrow Y$ be a continuous closed surjective function. Show that the topology on Y is the quotient topology defined by f.

4 Let (X, d) be a metric space. Prove that if every continuous function $f : X \longrightarrow \mathbf{R}$ is bounded (i.e. if for some $R > 0$ and all $x \in X$ we have $|f(x)| < R$), then X is compact.

5 Let X, Y be topological spaces, and $\mathcal{C}(X, Y)$ be the space of continuous functions $f : X \longrightarrow Y$ with the compact-open topology. Prove that if X has n points and has the discrete topology, then $\mathcal{C}(X, Y)$ is homeomorphic to Y^n with the product topology.

6 State the van Kampen theorem and use it to calculate $\pi_1(X)$, where X is the wedge of the two-sphere and the circle, i.e. X is homeomorphic to

$$\{(x, y, z) \in \mathbf{R}^3 \mid (x+1)^2 + y^2 + z^2 = 1\} \cup \{(x, y, z) \in \mathbf{R}^3 \mid z = 0 \text{ and } (x-1)^2 + y^2 = 1\}.$$

7 Suppose X is a path connected, locally path connected space with finite fundamental group. Prove that all maps $X \to S^1$ are nullhomotopic.

8 Suppose $U \subset \mathbf{R}^2$ is an open set and $x \in U$. Show that $U - \{x\}$ is not simply connected.