DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
January 6, 2012

Solve as many problems as you can. Partial solutions will be considered, but complete solutions will be graded higher. To pass the prelim you must solve at least one problem from each part.

Part I

1 Let X be a topological space. Prove that the following are equivalent:

(a) X is normal.
(b) Points in X are closed and if $A, B \subset X$ are closed, U open and $A \cap B \subset U$, then there are open sets $V, W \subset X$ such that

$$A \subset V, \quad B \subset W, \quad V \cap W \subset U.$$

2 Let X and Y be locally compact Hausdorff spaces with one-point compactifications $\tilde{X}$ and $\tilde{Y}$. Show that there is a quotient map from $\tilde{X} \times \tilde{Y}$ onto the one-point compactification of $X \times Y$.

3 Let W denote the following union of three sets in $\mathbf{R}^2$:

$$W = \{(t, 0) \mid 0 < t \leq 1\} \cup \left\{ \left(\frac{1}{n}, u\right) \mid n = 1, 2, 3 \ldots \text{ and } 0 \leq u \leq 1 \right\} \cup \left\{ \left(0, \frac{1}{2}\right) \right\}.$$

(a) Is W connected?
(b) Is W locally connected?

Prove your assertions.

4 Let X, Y, Z be topological spaces, $f : X \longrightarrow Y$ a continuous map. Let $\mathcal{C}(X, Z)$ and $\mathcal{C}(Y, Z)$ be the spaces of continuous maps $X \longrightarrow Z$ and $Y \longrightarrow Z$ respectively with the compact-open topology. Prove that the map

$$f^* : \mathcal{C}(Y, Z) \longrightarrow \mathcal{C}(X, Z)$$

defined by $f^*(g) = g \circ f$ is continuous.

Part II

5 Let $A \subset S^n$ be any proper subspace and let $f : A \longrightarrow X$ be a continuous map. Prove that if f extends to a continuous map $g : S^n \longrightarrow X$, then f is null-homotopic.

6 Suppose X is a topological space with a contractible universal cover. Show that every continuous map $S^n \longrightarrow X$ $(n \geq 2)$ is null-homotopic.

7 Give an example of a space whose fundamental group is a cyclic group of order six. Prove that the fundamental group of the space you constructed is cyclic.

8 Let X be the wedge $P^2 \vee P^2$ of two copies of the projective plane P^2, i.e. the space obtained from the disjoint union $P^2 \amalg P^2$ by identifying two points, one from each copy of P^2. Prove that the fundamental group of X is an infinite non-abelian group.