DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
January 7, 2013

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1 Let X be a topological space, and let X_0 be the set X with the finite complement topology. Show that the identity map from X to X_0 is continuous if and only if X is a T_1 space.

2 Let X be a topological space . Consider the following conditions:

(A) Every family (U_α) of pairwise disjoint open sets $U_\alpha \subset X$ is countable.

(B) There is a countable dense set $S \subset X$.

Show that (B) implies (A) but, in general, (A) does not imply (B).

3 For two topological spaces X and Y, let $\mathcal{C}(X, Y)$ be the space of continuous functions $f : X \longrightarrow Y$ equipped with the compact-open topology. Let $g : Y \longrightarrow Z$ be a continuous map to a third space Z. Prove that the map

$$\Phi : \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X, Z),$$

defined by $\Phi(f) = g \circ f$ is continuous.

4 Let f be a continuous one-to-one function from the unit interval into $\mathbf{R}^n$. Prove that if $n \geq 2$, then the image of f has no interior.

5 Let X be a space and $x_0 \in X$ be any point. We define a map

$$\Phi : \pi_1(X, x_0) \longrightarrow [S^1, X]$$

as follows: Let $f : [0,1] \longrightarrow X$ be a loop at x_0, i.e. $f(0) = f(1) = x_0$. Then there is a unique continuous map $g : S^1 \longrightarrow X$ satisfying $f = g \circ p$, where $p : [0,1] \longrightarrow S^1$ is the standard map $p(t) = (\cos 2\pi t, \sin 2\pi t)$. The map Φ is then defined by $\Phi([f]) = [g]$.

Prove that $\Phi(\alpha) = \Phi(\beta)$ if and only if α and β are conjugate in $\pi_1(X, x_0)$.

6 Let P^2 be the projective plane. Show that every covering map $f : P^2 \longrightarrow P^2$ is a homeomorphism.

7 A *knot* is a simple closed curve K in $\mathbf{R}^3$. Regard S^3 as a one-point compactification of $\mathbf{R}^3$. Prove that if K is a knot, the fundamental groups $\pi_1(\mathbf{R}^3 \setminus K)$ and $\pi_1(S^3 \setminus K)$ are isomorphic.

8 Consider the torus $T^2 = S^1 \times S^1 \subset \mathbf{C} \times \mathbf{C} = \mathbf{C}^2$. Let $F : T^2 \longrightarrow T^2$ be the map

$$F(z_1, z_2) = (z_1^4 z_2^2, z_1^6 z_2^3).$$

(a) Describe the homomorphism $F_* : \pi_1(T^2) \longrightarrow \pi_1(T^2)$ induced by F.
(b) Is F homotopic to a covering map?