DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
January 9, 2015

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1 Let $X = \{\text{prime numbers}\} \cup \{0\}$. We declare $A \subset X$ to be closed if either (i) A is empty, or (ii) $A = X$, or (iii) $0 \notin A$ and A is finite.

(a) Show that this defines a topology on X.
(b) Show that X is compact.

2 Let M be an n-manifold. Suppose that $C \subset U \subset M$, where C is connected and U is open. Show that for any two points $x, y \in C$, there is a path in U from x to y.

3 Let $F : X \times [0,1] \longrightarrow Y$ be a homotopy and $W \subset Y$ be an open set containing $F(X \times \{0\})$. Prove that if X is compact, then there is an $\varepsilon > 0$ such that $F(X \times [0, \varepsilon)) \subset W$. Show that if X is not required to be compact, the statement is no longer true.

4 For two spaces X, Y, let Y^X be the space of continuous maps $X \longrightarrow Y$ with the compact-open topology. Let X, Y, A, B be any four spaces, A and B locally compact Hausdorff. Prove that the map

$$\pi : X^A \times Y^B \longrightarrow (X \times Y)^{A \times B}$$

given by

$$\pi(f, g) = f \times g$$

is continuous.

5 Let $x \neq y$ be two points in the torus T^2. Compute the fundamental group $\pi_1(T^2 - \{x, y\})$.

6 Let $X = (S^1 \times I)/(S^1 \times \partial I)$. Show that $\pi_1(X) \cong \mathbf{Z}$.

7 Let $p : E \longrightarrow B$ be an n-fold covering map. If B is compact, show that E is also compact.

8 Find two examples of connected two-sheeted covers of the torus that are not equivalent.