DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
January 4, 2019

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1 Let $X = \{a, b, c, d\}$, with open sets $\emptyset$, $\{a, b\}$, $\{c\}$, $\{a, b, c\}$, and $\{a, b, c, d\}$, and let $Y = \{0, 1\}$ with open sets $\emptyset$, $\{0\}$, and $\{0, 1\}$. Show that the function $f : X \longrightarrow Y$ defined by $f(a) = f(b) = 0$ and $f(c) = f(d) = 1$ is a **quotient map** that is **not** an open mapping.

2 Let $f : X \longrightarrow Y$ be a closed, continuous, surjective map such that $f^{-1}(y)$ is a compact subspace of X for each $y \in Y$. Show that if Y is compact, then so is X.

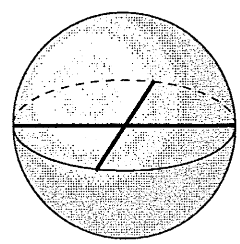
3 Write $\mathrm{Comp}(X)$ for the set of connected components of X.

(a) Show that a homeomorphism $X \xrightarrow{\cong} Y$ induces a bijection $\mathrm{Comp}(X) \xrightarrow{\cong} \mathrm{Comp}(Y)$.

(b) Let G be a topological group, and consider the function $c : G \longrightarrow \mathrm{Comp}(G)$ which takes each point to its connected component. Show that the group structure on G induces a group structure on $\mathrm{Comp}(G)$ for which c is a group homomorphism.

4 Let F be a finite set, equipped with the discrete topology, and let Y be any space. Show that the compact-open topology on $\mathrm{Map}(\cancel{X}F, Y)$ agrees with the product topology $\prod_F Y$.

5 Let X be the union of a sphere (S^2) and two diameters, as in the graphic to the right. Compute the fundamental group of X.



6 Describe all coverings of $\mathbb{RP}^2 \times \mathbb{RP}^2$ up to isomorphism.

7 Let $f : S^1 \longrightarrow S^1$ be a degree three map, and let $X = S^1 \cup_f D^2$ be the result of attaching a 2-cell to S^1 along the map f. Show that X is not a manifold.

8 Let $f : S^1 \longrightarrow S^1$ be an odd map (i.e. $f(-x) = -f(x)$) such that $f(x_0) = x_0$ for some $x_0 \in S^1$. Show that $f_* : \pi_1(S^1, x_0) \longrightarrow \pi_1(S^1, x_0)$ is multiplication by some **odd** integer.