DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
January 8, 2020

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1 Let $\mathcal{B}$ be the collection of subsets of $\mathbb{R}$ of the following two forms:

- sets of the form $(-b, -a) \cup (a, b)$, where $0 < a < b$
- sets of the form $(-\infty, -d) \cup (-c, c) \cup (d, \infty)$, where $0 < c < d$.

(a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$

(b) To which point(s) in $\mathbb{R}$ does the sequence $x_n = 1 - \frac{1}{n}$ converge?

2 Let Z be the subspace $(\mathbb{R} \times 0) \cup (0 \times \mathbb{R})$ of $\mathbb{R}^2$. Define $g\colon \mathbb{R}^2 \longrightarrow Z$ by

$$g(x, y) = \begin{cases} (x, 0) & \text{if } x \neq 0 \\ (0, y) & \text{if } x = 0. \end{cases}$$

(a) Is g continuous, when Z is equipped with the subspace topology?

(b) Show that in the quotient topology induced by g, the space Z is not Hausdorff.

3 Let G be a topological group acting on a space X. Show that if both G and X/G are connected, then X is connected.

4 Let $C(X, Y)$ denote the set of continuous maps $X \longrightarrow Y$. Let $M(X, Y)$ denote the set $C(X, Y)$ equipped with *some* topology for which the evaluation map

$$M(X, Y) \times X \longrightarrow Y$$

is continuous. Also, let $\mathrm{Map}(X, Y)$ denote $C(X, Y)$ equipped with the compact-open topology. Show that the identity map $M(X, Y) \longrightarrow \mathrm{Map}(X, Y)$ is continuous.

5 A subset $A \subset \mathbb{R}^n$ is *star convex* if there is a point $a_0 \in A$ so that for all $x \in A$, the line segment from a_0 to x is entirely contained in A.

(a) Show that any star convex subset is simply connected.

(b) Show that if α and β are paths in a star convex set such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$, then α and β are path-homotopic.

6 Define X to be the quotient of S^2 by imposing an antipodal relation, but only on the *equator*, so that $(x, y, 0) \sim (-x, -y, 0)$. Compute $\pi_1(X)$.

(OVER)

7 Let K denote the Klein bottle.

(a) Find $\pi_1(K)$.

(b) Let X be a connected and locally path-connected space such that $\pi_1(X) \cong \mathbb{Z}/3$. Show that there does not exist a covering map $X \longrightarrow K$.

8 (a) Show that S^3 is not homeomorphic to $S^2 \times S^1$.

The Hopf fibration $\eta\colon S^3 \longrightarrow S^2$ is the quotient map for a free S^1-action on S^3. (So $\eta^{-1}(x)$ is homeomorphic to S^1 for all $x \in S^2$.)

(b) A section for η is a map $u\colon S^2 \to S^3$ so that $\eta \circ u = \mathrm{id}_{S^2}$. Show that a section u would define a continuous bijection

$$S^2 \times S^1 \to S^3$$

by $(x, z) \mapsto u(x)z$. (Here S^1 acts on $S^3 \subset \mathbb{C}^2$ via complex multiplication.)

(c) Conclude that the Hopf fibration does not have a section.