Winter 2021 Topology Preliminary Exam

1. Let $U \subset \mathbb{R}^n$ be a bounded, nonempty, open subset. Show that the quotient space $\mathbb{R}^n/U$ is not Hausdorff.

2. Show that if A and B are compact subspaces of a space X, then so is $A \cup B$. If in addition, X is Hausdorff, show that $A \cap B$ is compact.

3. The configuration space of k points in $\mathbb{R}^2$, denoted $C(k, 2)$, is defined to be the collection of all ordered tuples $(x_1, \ldots, x_k)$ where $x_i \in \mathbb{R}^2$ for all i and $x_i \neq x_j$ when $i \neq j$. The topology on this space comes from the fact that it is a subset of $(\mathbb{R}^2)^k$.

   (a) Produce a homeomorphism $C(1, 2) \cong \mathbb{R}^2$.

   (b) Produce a homeomorphism $C(2, 2) \cong \mathbb{R}^2 \times (0, \infty) \times S^1$

4. For $x \in I = [0, 1]$, let

$$U_x = S\left(\{x\}, \left(x - \frac{1}{4}, x + \frac{1}{4}\right) \cap I\right)$$

   where $S(C, U)$ is the set of continuous functions $f \colon I \to I$ so that $f(C) \subset U$.

   (a) Show that the $\{U_x\}_{x \in I}$ are an open cover of $C(I, I)$ with the compact open topology.

   (b) Let $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}_{x_1 < x_2 < \cdots < x_n}$ be any finite collection of the U_x. Construct a continuous map $I \to I$ not in $U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}$

5. Let $T^2 = S^1 \times S^1$ denote the torus, and $\mathbb{R}P^2 = S^2/\sim$, where $\sim$ is the relation $v \sim -v$ on S^2, denote the real projective plane.

   (a) Prove that every continuous map $\mathbb{R}P^2 \to T^2$ is homotopic to a constant map.

   (b) Show that there is no covering map $T^2 \to \mathbb{R}P^2$.

6. Give explicit descriptions of two non homeomorphic, connected, two-sheeted covers of $\mathbb{R}P^2 \vee \mathbb{R}P^3$.

7. (a) Describe a connected space X such that $|\pi_1(X, x_0)| = 10$. (No proof necessary for this part.)

   (b) For X as in part a), suppose that $\tilde{X}$ is a connected space and that $p \colon \tilde{X} \to X$ is a covering map. Is it possible that $|p^{-1}(x_0)| = 6$? Justify your answer

8. Suppose $X_1, \ldots, X_m$ are convex open subsets of $\mathbb{R}^n$ such that the triple intersection $X_i \cap X_j \cap X_k$ is nonempty for all i, j, k.

   (a) Show that $(X_1 \cup \ldots \cup X_{m-1}) \cap X_m$ is path connected and nonempty.

   (b) Show that the fundamental group of $X_1 \cup \ldots \cup X_m$ is trivial. (Hint: use induction and part (a)!)