Topology Preliminary Exam
January, 2023

On grading: A necessary condition to pass this exam is to completely solve one point-set question and one of the more algebraic questions. An excellent exam will completely solve five problems and have some partial credit on the remaining questions.

1. A function (not necessarily continuous) $f\colon X \to Y$ is called proper if for any compact $K \subseteq Y$, $f^{-1}(K)$ is compact in X.

a. Give an example of a continuous function that is not proper.

b. Show that any proper function from a Hausdorff space X to a compact space Y is continuous.

2. Let $f\colon X \to Y$ be a quotient map. Assume that Y is connected and that each fiber $f^{-1}(y)$ for $y \in Y$ is connected as a subspace of X. Show that X is connected.

3. Let $S = \{a, b\}$ equipped with the generic point topology around a (a nonempty subset is open if it contains a). Let X be another space equipped with the generic point topology around $x \in X$. Note that X is based at x.

a. Show that a continuous map $f\colon S \to X$ satisfies $f(a) = x$ or $f(a) = f(b)$.

b. Produce a bijection (of sets) $C(S, X) \to X \vee X$, where $X \vee X$ is the wedge sum (viewed as a set). Here $C(S, X)$ is the set of continuous maps.

c. Let $\mathrm{Map}(S, X)$ be the set of continuous maps from S to X equipped with the compact open topology. Do the topologies on $\mathrm{Map}(S, X)$ and $X \vee X$ coincide?

d. Is $\mathrm{Map}(S, X)$ Hausdorff?

4. Assume that X is a path connected space and $x_0, x_1, x_2 \in X$. Prove that $\pi_1(X, x_2) \cong 0$ if and only if any two paths from x_0 to x_1 are homotopic through paths from x_0 to x_1.

5. Let W be the quotient of S^2 defined by imposing the (antipodal) relation $x \sim -x$ on the equator and also on a great circle (a longitudinal circle). Find the fundametal group of W.

6. Let G be a finite group acting freely on S^3 and let $Z = S^3/G$. Show that any continuous map from Z to a graph (i.e. a 1-dimensional CW-complex) must be null-homotopic.

7. a. Let K be the Klein bottle and let $x \in K$ be a point. Find $\pi_1(K, x)$.

b. Use the Hurewicz theorem to compute the homology group $H_1(K, \mathbb{Z})$.

c. Use that $H_n(K; \mathbb{Z}) = 0$ for $n \geq 2$ to compute the homology groups $H_i(K; \mathbb{Z}/2)$ for all $i \geq 0$.

8. Compute the relative homology group $H_*(X, A; \mathbb{Z})$ for the following pairs:

a. $X = S^2$, A is the equator.

b. X is the Mobius band, A is the boundary.