DEPARTMENT OF MATHEMATICS

TOPOLOGY PRELIMINARY EXAMINATION
JANUARY 3, 2024

INSTRUCTIONS: A necessary condition to pass this exam is to completely solve one point-set question and one of the more algebraic questions. An excellent exam will completely solve five problems and have some partial credit on the remaining questions.

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1) Let $f\colon X \longrightarrow Y$ be a function. Show, from the definition of a continuous function, that f is continuous if and only if for every subset $A \subset X$ and every limit point p (i.e. accumulation point) of A, the image $f(p)$ belongs to $\overline{f(A)}$.

Note that you are NOT permitted to use the fact that f is continuous if and only if $f(\overline{A}) \subset \overline{(f(A)}$ for every A.

2) Show that if X can be equipped with a CW structure using only finitely many cells, then X is compact.

3) For spaces W and Z, let $\mathrm{Map}(W, Z)$ denote the space of continuous maps $W \to Z$, equipped with the compact-open topology.

Suppose that A and B are locally compact and Hausdorff, and let X and Y be any spaces. Show that the map

$$P\colon \mathrm{Map}(A, X) \times \mathrm{Map}(B, Y) \longrightarrow \mathrm{Map}(A \times B, X \times Y)$$

defined by $P(f, g) = f \times g$ is continuous.

4) Find all covers (in the sense of covering spaces) of the Mobius strip.

5) Let $p\colon E \longrightarrow B$ be a covering, with E and B connected and locally path-connected, and pick a basepoint $e_0 \in E$. Show that the subgroup $p_*\pi_1(E, e_0) \leq \pi_1(B, p(e_0))$ is **normal** if and only if the group $\mathrm{Aut}_B(E)$ of deck transformations acts **transitively** on the fiber $F = p^{-1}(p(e_0))$.

6) Let X be the quotient of the square $I \times I$ with respect to the equivalence relation generated by $(t, 0) \sim (1, t/2)$ for all $t \in I$ and also $(0, 1) \sim (1, 1)$.

(a) Describe a CW structure on X

(b) Use the CW structure to find a presentation for $\pi_1(X)$ and show that this is a familiar group.

(OVER)

7) Suppose that X is a space that admits a cover by open subsets U and V such that U and V are both homotopy equivalent to S^2 and $U \cap V$ is homotopy equivalent to S^1. Calculate all of the homology groups of X.

8) Compute the singular homology groups of the 3-dimensional real projective space $\mathbb{RP}^3$. You **may** use knowledge of the homology groups $\mathrm{H}_*(\mathbb{RP}^2)$.