DEPARTMENT OF MATHEMATICS

TOPOLOGY PRELIMINARY EXAMINATION
JANUARY 10, 2025

INSTRUCTIONS: A necessary condition to pass this exam is to completely solve one point-set question and one of the more algebraic questions. An excellent exam will completely solve five problems and have some partial credit on the remaining questions.

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1) Consider the space $X = \mathbb{N} \cup \{x, y\}$ with open sets given by all subsets of $\mathbb{N}$ and also subsets S with the property that $S \cap \mathbb{N}$ is cofinite in $\mathbb{N}$.

(a) Show that X is not Hausdorff.

(b) Show that X is totally disconnected, meaning that the only nonempty connected subsets are singletons.

2) Let $f : X \longrightarrow Y$ be a closed, continuous, surjective map such that $f^{-1}(y)$ is a compact subspace of X for each $y \in Y$. Show that if Y is compact, then so is X.

3) Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of natural numbers. For each $i \in \mathbb{N}$, let X_i be a finite set, equipped with the discrete topology, and let $f_i \colon X_{i+1} \to X_i$ be a function. Let $L \subset \prod_{i \in \mathbb{N}} X_i$ be the subspace of the product (equipped with the product topology) consisting of tuples $(x_0, x_1, \dots)$ such that $f_i(x_{i+1}) = x_i$ for all i.

Show that L is closed in $\prod_i X_i$ and conclude that L is compact.

4) Let X be the space obtained by starting with a doughnut $S^1 \times D^2$ and removing 3 distinct points in the interior.

(a) Describe a CW structure on a space homotopy equivalent to X.

(b) Find $\pi_1(X, x)$ for any choice of $x \in X$.

5) Write F_n for the free group on n generators. Use covering space theory to describe a subgroup of F_2 that is isomorphic to F_5.

6) Let $f \colon S^1 \to S^1$ be the 5-sheeted cover and let $g \colon S^1 \to S^1$ be the 7-sheeted cover. Consider the non-connected covering space $f \amalg g \colon S^1 \amalg S^1 \to S^1$. For $b \in S^1$, describe the action of $\pi_1(S^1, b)$ on the preimage of b.

7) Let $f \colon S^1 \to S^1$ be the 3-sheeted cover, and let Y be the quotient $S^1 \times [0,1]/\sim$, where $(x, 0) \sim (f(x), 1)$ for all $x \in S^1$. Compute $H_0(Y)$ and $H_1(Y)$.

8) Let M be a compact, connected surface (2-manifold), and let $x \in M$ be a point. Find the relative homology group $\mathrm{H}_2(M, M - \{x\})$. (Hint: consider a neighborhood U of x such that U is homeomorphic to $\mathbb{R}^2$.)