**1** Define the *suspension* $SX$ of a topological space $X$ to be the quotient space of $X \times [0,1]$ obtained by identifying $(x,0)$ with $(x',0)$ and $(x,1)$ with $(x',1)$ for all $x, x' \in X$. Show that the $n$-dimensional sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$ is homeomorphic to the suspension $SS^{n-1}$ of the $(n-1)$-dimensional sphere $S^{n-1}$ for $n \geq 1$.

**2** Let $X$ be the subspace of $\mathbb{R}^2$ defined by

$$X = \{(0,0)\} \cup \{(1,0)\} \cup \bigcup_{n\geq 1} ([0,1] \times \{1/n\}).$$

Prove:

(a) The connected component of $(0,0)$ is the singleton $\{(0,0)\}$.
(b) Every set $A \subset X$ which is both open and closed and contains $(0,0)$ also contains $(1,0)$.

**3** Let $X = \prod_{\alpha \in J} X_\alpha$ be equipped with the product topology. Prove that if $X$ has a countable base, then

(a) Each $X_\alpha$ has a countable base.
(b) For all but countably many indices $\alpha$ the topology of $X_\alpha$ is indiscrete.

**4** Let $X$ be compact and let $\Delta = \{(x,x) | \, x \in X\}$ be the diagonal in $X \times X$. Prove that if the quotient space $(X \times X)/\Delta$ is metrizable, then so is $X$.

**5** Prove the following generalization of the Lebesgue number lemma. Let $A$ be a compact subset of a metric space $X$ and let $\mathcal{U}$ be an open cover of $A$ by open sets in $X$. Then there is an $\varepsilon > 0$ which satisfies the property that if $S$ is a subset of $X$ for which $S \cap A \neq \varnothing$ and $\mathrm{diam}(S) < \varepsilon$, then $S$ lies in some element of $\mathcal{U}$.

**6** Let $p : X \longrightarrow Y$ be a surjective map of topological spaces. Prove that the following are equivalent:

(a) $p$ is a quotient map.
(b) For every map $f : Y \longrightarrow Z$, $f$ is continuous if and only if $f \circ p$ is continuous.

**7** Let $p : E \longrightarrow B$ be a covering map, $B_0 \subset B$, $e_0 \in E$ and $p(e_0) = b_0 \in B_0$. Prove that if

$$\pi_1(B_0, b_0) \longrightarrow \pi_1(B, b_0)$$

is an isomorphism, then

$$\pi_1(p^{-1}(B_0), e_0) \longrightarrow \pi_1(E, e_0)$$

is also an isomorphism. (The arrows are inclusion-induced homomorphisms.)

**8** Let $X$ and $Y$ be respectively the Möbius band and the cylinder $S^1 \times [0,1]$.

(a) Prove that $X$ and $Y$ have the same homotopy type.
(b) Explain why $X$ and $Y$ are not homeomorphic.