1 Let X be a space in which all singletons are closed sets and which satisfies the following property: For every open set U and $p \in U$ there exists an open and closed set V such that $p \in V \subset U$. Prove that all components of X are singletons.

2 Let X be a locally compact space. Prove that the following three properties are equivalent:

(a) $X = \cup_{n=1}^{\infty} C_n$, where C_n is compact for $n = 1, 2, \ldots$
(b) $X = \cup_{n=1}^{\infty} C_n$, where C_n is compact and $C_n \subset \mathrm{Int}(C_{n+1})$ for $n = 1, 2, \ldots$
(c) X is Lindelöf, i.e. from any open covering of X we can choose a countable subcovering.

3 Let X be a metric space, let $D \subset X$ be dense, and let $X \longrightarrow Y$ be a continuous function for which $f|_D : D \longrightarrow Y$ is a homeomorphism. Prove that $D = X$.

4 Let A be a subspace of X which is a deformation retract of X. Prove that if $f, g : X \longrightarrow Y$ are continuous functions for which $f(a) = g(a)$, for all $a \in A$, then f is homotopic to g.

5 Let $p : E \longrightarrow B$ be a covering map and let $f : S^2 \longrightarrow E$ be a continuous function for which $pf : S^2 \longrightarrow B$ is null-homotopic. Prove that f is null-homotopic. You may find the general lifting theorem useful.

6 Let $f : S^2 \longrightarrow S^2$ be a continuous map such that $f(-x) \neq f(x)$ for all $x \in S^2$. Show that there is a continuous map $g : S^2 \longrightarrow S^2$ homotopic to f such that $g(-x) = -g(x)$ for all $x \in S^2$. [Hint: Take the midpoint of the shortest arc joining $f(x)$ and $-f(-x)$.]

7 Prove form the first principles that a closed unit interval is connected in the usual topology.

8 (a) Show that if X is a contractible space, then X is path connected and $\pi_1(X) = 0$.
(b) Give an example without proof of a non contractible path connected space with $\pi_1(X) = 0$.