Preliminary Exam – Topology – June 3, 2005

1. Let $I = [0,1]$ have the usual topology and let A be a subset of I. Let $\mathcal{T}_1$ be the subspace topology A inherits from I and let $\mathcal{T}_2$ be the order topology on A.

   (a) Prove that if A is a closed subset of I, then $\mathcal{T}_1 = \mathcal{T}_2$.

   (b) Find a subset A of I where $\mathcal{T}_1 \neq \mathcal{T}_2$.

2. Prove that there is no 1-1 continuous mapping of the unit circle S^1 into the real line R^1.

3. Let X be compact and let $\mathcal{F}$ be a family of continuous real-valued functions on X which satisfies the following properties.

   (a) If f and g are in $\mathcal{F}$, then their product fg is also in $\mathcal{F}$.

   (b) For each $x \in X$ there is a neighborhood U_x of x and an $f \in \mathcal{F}$ for which $f(U_x) = \{0\}$.

   Prove that $\mathcal{F}$ contains the function $f \equiv 0$.

4. Let X be a topological space and let $\sim$ be the equivalence relation defined on X by: $x \sim y$ iff y belongs to the component of X containing x.

   (a) Prove that if A and B are disjoint closed sets for which $A \cup B$ is saturated with respect to $\sim$, then so are both A and B.

   (b) Prove that the quotient space $X/\sim$ is totally disconnected.

5. Let A be the graph of the curve $y = \sin(1/x), 0 < x \leq 1$, and let B be an arc in the plane with endpoints $(0,0)$ and $(1, \sin(1))$ which intersects the closure of A, i.e. $\bar{A}$, in precisely these points. Prove that the space $X = \bar{A} \cup B$ is simply connected.

6. Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point.

   (a) Explain how X is obtained by attaching a 2-cell to S^1 by using a copy of S^1 arising from an arc in S^2 between the poles.

   (b) Use (a) to compute the fundamental group of X.

7. Let $p: E \to B$ be a simply connected covering space of B, let B' be a subset of B which is path connected and locally path connected, and let E' be a path component of $p^{-1}(B')$.

   (a) Prove that $p: E' \to B'$ is a covering space of B'.

   (b) Prove that $p: E' \to B'$ is the covering space corresponding to the kernel of the inclusion-induced homeomorphism $\pi_1(B') \to \pi_1(E')$.
   [Hint: Observe that each loop in the kernel comes from a loop in E.]