Preliminary Examination – Topology – June 1, 2006

1. A space X has the **fixed point property** provided that for each continuous function $f : X \to X$, there is a point $x \in X$ such that $f(x) = x$. Prove that if X has the fixed point property, then it is connected. Give an example of a connected space without the fixed point property.

2. A continuous function $f : X \to X$ is a **universal map on** X if for each continuous $g : X \to X$, $f(x) = g(x)$ for some $x \in X$. Prove that the universal maps on the unit interval I are precisely the continuous surjections.

3. Show that an uncountable product of unit intervals is not first countable in the product topology.

4. Let X be compact and Y Hausdorff, and let $C(X, Y)$ be the space of continuous functions from X to Y with the compact-open topology. Prove that the following subset C' of $C(X, Y)$ is closed:

$$C' = \{f \in C(X, Y) | f(X) = Y\}.$$

5. Let X be the quotient space obtained from the closed interval $[-1, 1]$ by identifying t with $-t$ for all t in the open interval $(-1, 1)$. Find two compact sets in X whose intersection is not compact.

6. Classify the letters of the word HOMOTOPY according to homotopy type and topological type.

7. Let X be locally connected, let $p : E \to B$ be a covering map, and let $f : X \to B$ be both continuous and open. Prove that any continuous lifting of f must also be open.

8. Compute the fundamental group of the space which is the union of the unit sphere $x^2 + y^2 + z^2 = 1$ and the xy-plane.