Topology Preliminary Exam

May 31, 2007

1. Compute the fundamental group of
   (a) The complement of the closed disk $x^2 + y^2 \leq 1$, $z = 0$ in R^3.
   (b) The solid torus obtained by rotating the open disk $x^2 + y^2 < 1$, $z = 0$ in R^3 about the line $y = 2$, $z = 0$.

2. Suppose that X and Y have the same homotopy type. Show that the path components of X and Y are in one-to-one correspondence.

3. Let X be a Hausdorff space and suppose $f: X \to X$ is continuous. Prove that the set
$$A = \{x \in X | f(x) = x \text{ or } f^2(x) = x\}$$
is closed in X.

4. Let $A \subset X$, a topological space.

   Prove or disprove:
   (a) $int(A) = int(int(A))$
   (b) $int(A) = int(cl(A))$
   (c) if A is finite, then $A = cl(A)$.

5. Prove or disprove: A retract of a contractible space is contractible.

6. Let R_r denote the set of real numbers equipped with the topology generated by the following basis.
$$B = \{[x, y) \,|\, x \text{ and } y \text{ are rational and } x < y\}$$
   (a) Use the Urysohn metrization theorem to prove that R_r is metrizable.
   (b) Is R_r homeomorphic with R?

7. Let X be any unbounded metric space. Prove that if X has a limit point, then the projection $p: X \times X \to X$ onto the first factor is not a closed map. Deduce that p is a closed map if and only if X is discrete.

8. Let $p: E \to B$ be a covering map such that E is path connected and locally path connected. Prove that p is inessential (i.e. null homotopic) iff E is contractible. You may use the homotopy lifting theorem.