DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
June 5, 2009

1 Let $X = \{1, 2, 3, \ldots\}$ have the finite complement topology

$$\mathcal{T} = \{\varnothing\} \cup \{A \mid A \subset X \text{ and } X - A \text{ is finite}\}.$$

Prove that any continuous function $f : X \longrightarrow \mathbf{R}$ must be constant.

2 The set $M(2, \mathbf{R})$ of all 2×2 matrices with real entries can be identified with $\mathbf{R}^4$ because there are 4 real-valued entries in the matrix. Consider the following subspace of $M(2, \mathbf{R})$:

$$D := \{A \in M(2, \mathbf{R}) : \det(A) \in [-3, 5]\}.$$

(a) Is D compact? Justify your answer.
(b) Is D path-connected? Justify your answer.

3 Let X be a topological space and suppose there is a $p \in X$ such that if $U \subset X$ is open and $p \in U$, then $U = X$. Prove that X is compact and path connected.

4 Let $f, g, h : X \longrightarrow Y$ be three continuous functions. Prove that if Y is Hausdorff, then the set

$$U = \{x \in X \mid f(x) \neq g(x) \neq h(x) \neq f(x)\}$$

is open.

5 Let $B^2 \subset \mathbf{R}^2$ be the open unit disc:

$$B^2 = \left\{x \in \mathbf{R}^2 \mid \|x\| < 1\right\}.$$

(a) Prove that if $f, g : B^2 \longrightarrow [0, 1]$ are uniformly continuous, then so is $fg : B^2 \longrightarrow [0, 1]$.
(b) Is the previous statement true for functions $f, g : B^2 \longrightarrow \mathbf{R}$?

6 Let $f : S^1 \longrightarrow X$ be a continuous function for which the induced homomorphism f_* of fundamental groups is trivial. Prove that f is nullhomotopic.

7 Compute the fundamental group of $X - \{x\}$ for each point x of the space $X = \mathbf{R} \times [0, 1]$.

8 Let $\alpha : S^2 \longrightarrow P^2$ be the continuous map that identifies x and $-x$ to a point.

(a) Is α a homotopy equivalence? Justify your answer.
(b) Is there another connected space X not equal to S^2 or P^2 such that $S^2 \longrightarrow X \longrightarrow P^2$, where both of the arrows indicate covering maps? Justify your answer.

(c) Is there another connected space Y not equal to S^2 or P^2 such that $Y \longrightarrow S^2 \longrightarrow P^2$, where both of the arrows indicate covering maps? Justify your answer.