DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
June 4, 2010

1 A topological space X is called *irreducible* if, whenever $X = F \cup G$ with F and G closed, then either $X = F$ or $X = G$. A subspace is irreducible if it is irreducible in the subspace topology. Show that if X is irreducible and $U \subset X$ is open, then U is irreducible.

2 Let $f : X \longrightarrow Y$ be continuous, where X is a complete metric space and Y is Hausdorff. Prove that if $\{A_1, A_2, \ldots\}$ is a descending sequence of closed sets in X for which $\operatorname{diam}(A_n) \to 0$, then $f(\cap A_n) = \cap f(A_n)$.

3 Let X be a connected, locally connected, locally compact Hausdorff space. Prove that for every $a, b \in X$, there is a compact connected subspace $Y \subset X$ such that $a, b \in Y$.

4 Let X and Y be metric spaces with metrics d_X and d_Y respectively. Define a metric D on $X \times Y$ by

$$D[(x_1, y_1), (x_2, y_2)] = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}.$$

Show that the topology induced by this metric is the same as the product topology.

5 Let S be a countable set and X be a compact metric space. Show that every sequence $\{f_n\}$ of maps $f_n : S \longrightarrow X$ has a point-wise convergent subsequence.

6 Describe the covering spaces for the torus $S^1 \times S^1$ corresponding to the following subgroups of $\pi_1(S^1 \times S^1) = \mathbf{Z} \times \mathbf{Z}$:

(a) the subgroup generated by $m \times 0$, where m is a positive integer;
(b) the trivial subgroup;
(c) the subgroup generated by $m \times 0$ and $0 \times n$, where m and n are positive integers.

7 You are given a covering map $p : E \longrightarrow B$ and continuous functions

$$f : (B, b_0) \longrightarrow (B, b_1), \quad F : E \longrightarrow E$$

for which $p \circ F = f \circ p$. Prove that if E is simply connected, then $f(p^{-1}(b_0))$ is a singleton if and only if $F_* : \pi_1(B, b_0) \longrightarrow \pi_1(B, b_1)$ is the trivial homomorphism.

8 Let M be an n-manifold, $n \geq 3$. Prove that for every $m \in M$ and every $x_0 \neq m$, the groups $\pi_1(M, x_0)$ and $\pi_1(M - \{m\}, x_0)$ are isomorphic.