DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
June 6, 2011

1 Let $X = \mathbf{R}$ and let $\mathcal{U}$ be the collection of all sets $U \subset \mathbf{R}$ so that either $U = \mathbf{R}$ or $U \cap [0,1] = \varnothing$. Prove that $(X, \mathcal{U})$ is compact.

2 Let $p : \tilde{X} \longrightarrow X$ be a covering map with $p^{-1}(x)$ finite for all $x \in X$.

(a) Show that $\tilde{X}$ is Hausdorff if and only if X is Hausdorff.
(b) Show that $\tilde{X}$ is compact if and only if X is compact.

3 Let $h : Y \longrightarrow Z$ be a continuous map and for every X, let $h_\sharp : \mathcal{C}(X,Y) \longrightarrow \mathcal{C}(X,Z)$ be the map $h_\sharp(f) = h \circ f$. Here $\mathcal{C}(X,Y)$ and $\mathcal{C}(X,Z)$ are the sets of continuous maps $X \longrightarrow Y$ and $X \longrightarrow Z$ respectively. Prove that if $h_\sharp$ is a bijection for all X, then h is a homeomorphism.

4 Let X be a locally compact Hausdorff space. Prove that if the connected components of X are all non-compact, then the one-point compactification of X is connected.

5 For $a, b \in \mathbf{Z}$ with $b > 0$ define the set $N_{a,b} = \{a + nb \mid n \in \mathbf{Z}\}$.

(a) Identify the following sets: $N_{0,1}$, $N_{0,2}$, and $N_{1,2}$.
(b) We define a topology, $\mathcal{P}$, on $\mathbf{Z}$ as follows: U is open if either (i) $U = \varnothing$, or (ii) for every $a \in U$ there is some $b > 0$ so that $N_{a,b} \subset U$. Show that $\mathcal{P}$ is a topology.
(c) Show that each $N_{a,b}$ is closed.
(d) Prove that there is no finite non-empty set that is open.
(e) Let $\mathbf{P}$ be the set of prime integers. Show that

$$\mathbf{Z} \setminus \{-1, 1\} = \bigcup_{p \in \mathbf{P}} N_{0,p}.$$

(f) Show that $\mathbf{P}$ is infinite by contradiction.

6 Let X be a topological space and A be a subspace of X. The pair (X, A) has the *homotopy extension property* with respect to a space Y if each continuous function

$$f\colon (X \times 0) \cup (A \times I) \to Y$$

extends to a continuous function

$$F : X \times I \longrightarrow Y.$$

Show (X, A) has the homotopy extension property with respect to every space Y if and only if $(X \times \{0\}) \cup (A \times I)$ is a retract of $X \times I$.

7 Let X be the square $[0,1] \times [0,1]$, and let Y be the quotient space obtained by identifying the four corners (i.e., $\{(0,0), (0,1), (1,1), (1,0)\}$ is one equivalence class and the remaining equivalence classes are singletons.) Prove that Y is not simply connected.

8 Construct a simply-connected covering space of the space $X \subset \mathbf{R}^3$ that is the union of a sphere and a diameter. Carefully describe the covering map and prove that the covering space is simply connected.