DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
May 31, 2012

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1 Let X be the set of continuous functions $f : [a, b] \longrightarrow \mathbf{R}$. Consider the metric d^* on X given by
$$d^*(f, g) = \int_a^b |f(t) - g(t)|\, dt,$$
for $f, g \in X$ and let d be the usual metric on $\mathbf{R}$. For each element $f \in X$, define
$$I(f) = \int_a^b f(t)\, dt.$$
Prove that this function $I : (X, d^*) \to (\mathbf{R}, d)$ is continuous.

2 Prove that among the four spaces
$$[0, 1) \times (0, 1),\ [0, 1) \times [0, 1),\ [0, 1] \times (0, 1),\ [0, 1] \times [0, 1]$$
there is exactly one pair of homeomorphic spaces.

3 A space X is a *Lindelöf space* if each open cover has a countable subcover.

(a) Prove the continuous image of a Lindelöf space is Lindelöf.
(b) Prove that $\mathbf{R}$ with the usual topology is Lindelöf.
(c) Is $\mathbf{R}$ with the finite complement topology Lindelöf?

4 Let X be a T_1-space (i.e. points are closed in X). Prove that the following are equivalent:

(a) X has the discrete topology;
(b) X is first countable (every point has a countable base of open neighborhoods) and every compact subspace of X is finite.

5 Let $\mathcal{C}(X, Y)$ be the set of continuous functions from X to Y, with the compact-open topology.

(a) Show that $\mathcal{C}(X, Y \times Z)$ is homeomorphic to $\mathcal{C}(X, Y) \times \mathcal{C}(X, Z)$.
(b) Is $\mathcal{C}(X \times Y, Z)$ homeomorphic to $\mathcal{C}(X, Z) \times \mathcal{C}(Y, Z)$?

6 Let X be the Euclidean 3-space $\mathbf{R}^3$ with two lines deleted as follows:
$$X = \mathbf{R}^3 - (\{(t, 0, 1) \mid t \in \mathbf{R}\} \cup \{(t, 1, 0) \mid t \in \mathbf{R}\}).$$
Compute the fundamental group of X.

7 Let X be a path connected, locally path connected space. Prove that if every map $f : S^1 \longrightarrow X$ is nulhomotopic, then every map $f : X \longrightarrow S^1$ is nulhomotopic. Show by example that the converse is not true.

8 Prove that there is no open cover $\{U, V\}$ of the real projective plane $\mathbf{R}\mathrm{P}^2$, in which U and V are contractible and $U \cap V$ is path connected.