DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
June 3, 2013

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1 (a) Show that for any spaces X and Y the projection map

$$p_X : X \times Y \longrightarrow X$$

is open.

(b) Show that if Y is compact, then p_X is a closed map.

(c) Give an example to show that without compactness of Y part (b) does not hold.

2 Let $f : X \to Y$ be a quotient map of topological spaces, such that Y is connected and each set $f^{-1}(y)$, $y \in Y$, is a connected subspace of X. Show that X is connected.

[Recall that a mapping $f : X \to Y$ of topological spaces is a quotient map if f is surjective and a subset U of Y is open if and only if $f^{-1}(U)$ is open in X.]

3 Let X be an unbounded locally compact metric space and $\hat{X}$ be its one-point compactification. If $f : X \longrightarrow \mathbf{R}$ is a continuous function, prove that the following are equivalent:

(a) f extends to a continuous function $\hat{f} : \hat{X} \to \mathbf{R}$;

(b) For every $\varepsilon > 0$, there is a compact $K \subset X$ such that for every $x, y \in X - K$, we have $|f(x) - f(y)| < \varepsilon$.

4 Let G be a topological group, and let $H \subset G$ be a subgroup.

(a) Show that the closure $\overline{H}$ is also a subgroup.

(b) Show that if H is an open subgroup, then it must also be closed.

5 Let X be a convex subset of $\mathbf{R}^n$. [A subset X of $\mathbf{R}^n$ is convex if $(1-t)\mathbf{x} + t\mathbf{y} \in X$ for all $\mathbf{x}, \mathbf{y} \in X$ and real numbers t satisfying $0 \le t \le 1$.]

(a) Prove that any two continuous functions mapping some topological space into X are homotopic.

(b) Prove that any two continuous functions mapping X into some path-connected topological space Y are homotopic.

6 Show that there is no covering of the torus by the Klein bottle.

7 Let $X = S^1 \times \mathbf{R} \subset \mathbf{C} \times \mathbf{R}$. Show that every homeomorphism $F : X \longrightarrow X$ is homotopic to either the identity map $(z, t) \mapsto (z, t)$ or to the map $(z, t) \mapsto (\bar{z}, t)$.

8 (a) Compute the fundamental group of the projective plane with one point removed.

(b) Compute the fundamental group of the projective plane with two points removed.