DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
June 9, 2014

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1 Consider the set $X = \mathbf{R}^\infty$ of sequences of real numbers, equipped with the *box* topology. Find a sequence $(\mathbf{x}_n)$ of points in X that converges *pointwise* to the constant sequence $\mathbf{0} = (0, 0, \dots)$ but which does not converge in the box topology.

2 Let $\mathrm{SL}(2, \mathbf{R}) \subset \mathbf{R}^4$ denote the space of 2×2 matrices of determinant 1, with the subspace topology. For any invertible 2×2 matrix A, consider the map

$$\mathrm{conj}_A : \mathrm{SL}(2, \mathbf{R}) \longrightarrow \mathrm{SL}(2, \mathbf{R})$$

defined by

$$\mathrm{conj}_A(X) = AXA^{-1}\,.$$

(a) Is conj_A continuous for all choices of A? Why or why not?
(b) Is conj_A a homeomorphism? Why or why not?

3 Recall that a family $\{X_\alpha\}_{\alpha \in J}$ of subspaces of a space X is *locally finite* if every point of X has a neighborhood meeting X_α for only finitely many indices $\alpha \in J$. Let $\{K_\alpha\}_{\alpha \in J}$ be a locally finite family of compact subspaces of a space X. Prove that $\bigcup_{\alpha \in J} K_\alpha$ is locally compact in the subspace topology.

4 Prove that all metric spaces are normal.

5 Consider $S^1 \subset \mathbf{C}$ with basepoint $1 \in S^1$. On the set of continuous based maps $S^1 \longrightarrow S^1$, consider the following operations:

(a) Path-composition, given by

$$\alpha\beta(z) = \begin{cases} \alpha(z^2) & \text{if } \mathrm{Im(z)} \geq 0 \\ \beta(z^2) & \text{if } \mathrm{Im(z)} \leq 0 \end{cases}$$

(b) Point-wise multiplication, given by $\alpha \cdot \beta(z) = \alpha(z)\beta(z)$.
(c) Function composition, given by $\alpha \circ \beta(z) = \alpha(\beta(z))$.

Determine the operations induced on $\pi_1(S^1, 1) \cong \mathbf{Z}$ by the above three operations.

6 Let X and Y be two homotopy equivalent path connected spaces, and $x_0 \in X, y_0 \in Y$. Prove that $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

7 Let X be the complement of three distinct points in $\mathbf{R}^4$.

(a) Describe a 2-dimensional CW complex $Y \subset X$ that is a deformation retract of X and explain why it is a deformation retract.
(b) Determine the fundamental group of X.

8 Show that the only compact, connected surface M that can cover the torus T^2 is T^2 itself.