DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
June 1, 2015

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1 Find all connected, finite metric spaces.

2 Let $I = [0,1]$ and $O = (0,1)$ be the closed and open unit intervals, respectively. Let $X = (I \times I)/(O \times O)$.

(a) Show that X is not Hausdorff.
(b) Is X homeomorphic to $(I/O) \times (I/O)$?
(c) Consider the constant sequence at $(1/2, 1/2)$. To what point(s) does the sequence converge?

3 Let $X = \mathbf{R} \times \{0,1\}$, where $\mathbf{R}$ has the usual topology and $\{0,1\}$ has the trivial (indiscrete) topology. Find two compact sets $A, B \subset X$ such that $A \cap B$ is *not* compact.

4 For two spaces X, Y, we write $\mathrm{Map}(X,Y)$ for the set of continuous maps $X \longrightarrow Y$ equipped with the compact-open topology.

Let Q be the quotient space obtained from collapsing a non-empty closed subspace A of a compact Hausdorff space X to a point, and let $\pi : X \longrightarrow Q$ be the quotient map. Prove that the image of the induced map $\pi^\star : \mathrm{Map}(Q,Y) \longrightarrow \mathrm{Map}(X,Y)$ defined by $\pi^\star(g) = g \circ \pi$ consists of the maps $f : X \longrightarrow Y$ for which $f(A)$ is a single point.

5 Show that every homomorphism $\pi_1(S^1) \longrightarrow \pi_1(S^1)$ is of the form $\phi_\star$ for some $\phi : S^1 \longrightarrow S^1$.

6 Let $x \neq y$ be two points of a compact connected surface M. Show that $M - \{x,y\}$ is *not* simply-connected.

7 Find all two-fold connected coverings of the Klein Bottle.

8 The *double mapping cylinder* of maps $f : A \longrightarrow X$ and $g : A \longrightarrow Y$ is the quotient of the disjoint union $X \amalg (A \times I) \amalg Y$ under the relation generated by $f(a) \sim (a,0)$ and $g(a) \sim (a,1)$.

(a) Compute the fundamental group of the double mapping cylinder of the degree 3 and degree 5 maps $S^1 \longrightarrow S^1$.
(b) Compute the fundamental group of the double mapping cylinder of the degree 4 and degree 6 maps $S^1 \longrightarrow S^1$.