# DEPARTMENT OF MATHEMATICS

## Topology Preliminary Examination
## June 3, 2016

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1 Let $X$ be a topological space and suppose there is a $p \in X$ such that if $U \subset X$ is open and $p \in U$, then $U = X$. Show that $X$ is compact and path-connected.

2 For two topological spaces $X, Y$, let $\mathcal{C}(X, Y)$ be the space of continuous functions $X \longrightarrow Y$ with the compact-open topology. Prove that if $Y$ is Hausdorff, then so is $\mathcal{C}(X, Y)$.

3 Let $f : X \times C \longrightarrow \mathbf{R}$ be continuous. Assume that $C$ is compact and set

$$g(x) = \max\{f(x, c) \mid c \in C\}.$$

(Since $C$ is compact, there is a maximum.) Show $g : X \longrightarrow \mathbf{R}$ is continuous.

4 Given a continuous map $f : A \longrightarrow Y$, where $A$ is a subspace of a space $X$, prove that $f$ can be extended over $X$ if and only if $Y$ is a retract of $X \cup_f Y$.

5 Let $f : S^2 \longrightarrow S^2$ be a continuous function without fixed points. Prove that $f$ is surjective.

6 Let $M_1$ and $M_2$ be connected $n$-manifolds and let $M_1 \# M_2$ be their connected sum. Prove that if $n \geq 3$, then $\pi_1(M_1 \# M_2)$ is the free product of $\pi_1(M_1)$ and $\pi_1(M_2)$.

7 Let $X$ be a space, and let $a = (1, 0) \in S^1 \subset \mathbf{R}^2$ and

$$A = (S^1 \times \{a\}) \cup (\{a\} \times S^1) \subset S^1 \times S^1 = T^2.$$

Prove that the following are equivalent:

(a) For every $x_0 \in X$, $\pi_1(X, x_0)$ is abelian.
(b) Every continuous map $f : A \longrightarrow X$ extends to a continuous map $g : T^2 \longrightarrow X$.

8 Let $p : \tilde{G} \to G$ be a continuous homomorphism of topological groups such that $(\tilde{G}, p)$ is a path-connected covering space of $G$. Let $K$ denote the kernel of $p$.

(a) Show $K$ is a discrete subspace of $\tilde{G}$ which is contained in the center of $\tilde{G}$.
(b) For each element $k \in K$, define a map $\phi_k : \tilde{G} \to \tilde{G}$ by $\phi_k(x) = xk = kx$. Prove the map $k \mapsto \phi_k$ is an isomorphism from $K$ to the group of deck (or covering) transformations of $(\tilde{G}, p)$.