DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
June 6, 2018

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1 Show that for every continuous map $f : X \longrightarrow Y$ there is a space Z, a quotient map $q : X \longrightarrow Z$, and an injective (continuous) map $i : Z \longrightarrow Y$ such that $f = i \circ q$.

2 Let X be a locally compact Hausdorff space. Prove that if X has no compact connected components, then the one-point compactification of X is connected.

3 Let $f : X \longrightarrow Y$ be a quotient map of topological spaces such that Y is connected and each fiber $f^{-1}(y)$, $y \in Y$, is a connected subspace of X. Show that X is connected.

Find a counterexample if f is only assumed to be a continuous surjection.

4 Let X be a G-space, where G is a discrete group. If H is a subgroup of G, define

$$X^H = \{x \in X \mid hx = x \text{ for all } h \in H\}.$$

Give X^H the subspace topology.

Let $\mathrm{Map}_G(G/H, X)$ be the set of all G-equivariant maps $G/H \longrightarrow X$. Give this the subspace topology from the compact-open topology on the maps $G/H \longrightarrow X$.

Show that $\mathrm{Map}_G(G/H, X)$ and X^H are homeomorphic.

5 Let X be obtained by glueing together two copies of S^2, where the glueing identifies the two south poles and separately identifies the two north poles. Find $\pi_1(X)$.

6 Let $p : S^n \longrightarrow \mathbf{R}\mathrm{P}^n$ $(n \geq 2)$ be the standard two-fold covering, and let $X \subset \mathbf{R}\mathrm{P}^n$. Prove that $p^{-1}(X)$ is path-connected if and only if X is path-connected and the inclusion $i : X \hookrightarrow \mathbf{R}\mathrm{P}^n$ induces a surjection $i_* : \pi_1(X, x_0) \longrightarrow \pi_1(\mathbf{R}\mathrm{P}^n, x_0)$ for all $x_0 \in X$.

7 Describe all coverings of $S^1 \vee S^2$ up to isomorphism.

8 Construct a connected (not necessarily finite) graph X and continuous endomorphisms $f, g : X \longrightarrow X$ so that $f \circ g$ is the identity map, but f and g do not induce isomorphisms on π_1.